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Analytical Investigation for the Displacement of Beams with Flexible Supports

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Method Article

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ABSTRACT

The closed form (analytical) solution for the displacement of a beam with semi-rigid supports under dynamic pulse loading has been developed. Essential (Dirichlet) boundary conditions are prescribed and the equation of motion and subsequent mixed (Robin) boundary conditions are derived using Hamilton's principle (principle of least action). Using the exact assumed modes for various semi-rigid supports, the temporal displacements (generalised coordinates) are obtained. The displacement field is derived as a series solution with each term being the product of a generalised coordinate and an exact shape function. The derived exact shape functions, which depend upon a set of dimensionless parameters, are obtained through an eigenvalue analysis and define the associated eigenfunctions of the generalised coordinates. A table is presented to aid easy formulation of exact modes for varies beams using an intrinsic non-dimensional parameter, α. Using Galerkin's weighted residual the equation of motion is transformed from a partial differential equation to an ordinary differential equation for easy calculations.

Keywords: Galerkin method; analytical solution; series expansion; eigenvalue; pulse load.

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1. INTRODUCTION

This paper presents an investigation of the transverse vibration in a beam with varying rotational support conditions (ranging from simply-supported to fixed connections) under impulsive loads. This type of problem arises in many engineering systems, which are over-simplified as either simply-supported or fixed-ended beams. Semi-rigid connections are, more often than not, encountered in buildings, continuous span bridge beams, portal seismic resistant frames, etc. An example of the application of beams with semi-rigid supports is in offshore blast walls. Researchers have attempted to develop numerical solutions for the transverse displacement for such idealised blast wallse.g.Hsu, Langdon and Schleyer [1,2]. Despite the fact that their methods yield results well-corroborated with experiment and numerical simulations, their works only provide an approximate solution as the modes of vibration employed are not exact and only fundamental modes are taken into account.

S.M. Han et al. [3] have developed solutions for transverse free vibration of beams using various beam theories viz. Euler-Bernoulli, Rayleigh, Shear and Timoshenko beam theories. Their work was limited to determining the natural frequencies and modes for beams with four sets of boundary conditions: free-free, clamped-clamped (built-in), hinged-hinged (simply supported), and clamped-free (propped).

Recently, Jovanovic [4] has shown the problems that arise in solving for the transverse displacement of dynamically loaded beams when an unconventional boundary condition sets in. He presented the generalised Fourier series solution for transverse vibration of a beam subjected to a viscous boundary condition. The model of the system produces non-self-adjoint eigenvalue-like problem, which does not yield orthogonal eigenfunctions. This condition renders the problem unsolvable using conventional methods. He employed the Hilbert space methods in dealing with this problem.

Haddadpour [5] used a relatively new decomposition method in analysing the vibration of an Euler-Bernoulli beam concentrating on traditional support conditions. This approach is called the Adomian decomposition [6,7] and in this regard a general approach based on the generalised Fourier series expansion is applied.

Interestingly, some researchers have used the very robust and versatile He's variational iteration method [8] and other extensions [9] of it in order to obtain the free vibration of Euler-Bernoulli beams with traditional support conditions. This technique helps in determining the beam's natural frequencies and mode shapes and a rapidly convergent series is obtained for the solution. Liu and Gurram [10] have shown that the results obtained are the same with those using Adomian decomposition. A further comparison of the two methods has been presented recently [11]. Regrettably, in these works novel techniques are used in conjunction with the mere conventional support conditions used by Han et al. [3].

This paper addresses the pitfalls of the various works conducted on the dynamic transverse behaviour of beams. The hitherto models adopted assume over simplistic, though sometimes-useful approximations of support conditions by assuming hinged, clamped or free end conditions. More practical and realistic boundary conditions are addressed in this paper. The current work provides a full solution of the transverse displacement through obtaining the eigenfunctions of a semi-rigidly supported beam and proposing a series solution for the transverse displacement function. A set of dimensionless parameters are extracted using Buckingham's Pi-theorem and are used in determining the eigenvalues and eigenmodes. A word or two has been added for possible truncation procedure in impulsively loaded regimes for engineering applications.

2. PROBLEM STATEMENT

Fig. 1 shows the schematic of a beam with semi-rigid rotational support conditions. These conditions replicate practical support conditions in most engineering applications. The model parameters defining the problem uniquely are the Young modulus E, second moment of inertia I, cross sectional area A, density of beam ρ, length of the beam $L$and rotational stiffness of support K_θ.

Using Hamilton's principle (equation (1)), the equation of motion is derived as equation (2).

$$\int_{t_1}^{t_2} \delta(T-U)dt + \int_{t_1}^{t_2} \delta W dt = 0 \tag{1}$$

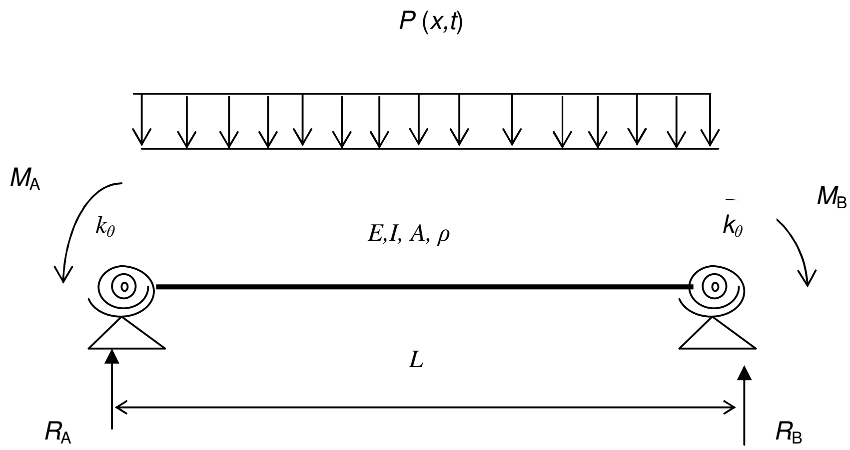

Fig. 1. Schematic model of a beam with semi rigid connections

$$\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = P(x,t)\ 0 < x < L \ \text{where} a^2 = \frac{EI}{\rho A} \tag{2}$$

Essential boundary conditions are prescribed in equation (3). Subsequently mixed boundary conditions are derived in equations (4) and (5).

$$w(0,t) = 0\,, \qquad w(L,t) = 0 \tag{3}$$

$$EIw''(L,t) + K_\theta w'(L,t) = 0 \tag{4}$$

$$-EIw''(0,t) + K_\theta w'(0,t) = 0 \tag{5}$$

3. EIGENVALUES AND EIGENFUNCTIONS (EIGENPROBLEM ANALYSIS)

The parameters λ_n and functions $\phi_n(x)$ which are analogous to natural frequencies and modes of the idealised structure, respectively, are determined by solving the self-adjoint eigenvalue problem in equation (6).

$$\phi'''' - \lambda^4 \phi = 0 \tag{6}$$

λ is the eigenvalue and the φ eigenfunction. The solution to equation (6) is the following eigenfunction:

$$\phi(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} + C_3 \sin(\lambda x) + C_4 \cos(\lambda x) \tag{7}$$

Satisfying boundary conditions and after some mathematical manipulation, the constant terms in equation (7) are given as:

$$C_2 = \frac{C_1\left(\left((\alpha^2 - 2\lambda^2 L^2 - \lambda L\alpha)\sin(\lambda L) + 3\alpha\cos(\lambda L)\left(\lambda L + \frac{\alpha}{3}\right)\right)e^{\lambda L} + \alpha(\sin(\lambda L)^2 + \cos(\lambda L)^2)(\lambda L - \alpha)\right)}{\left((2\lambda^2 L^2 - \lambda L\alpha - \alpha^2)\sin(\lambda L) - 3\left(\lambda L - \frac{\alpha}{3}\right)\alpha\cos(\lambda L)\right)e^{-\lambda L} - \alpha(\sin(\lambda L)^2 + \cos(\lambda L)^2)(\lambda L + \alpha)}$$

$$C_3 = \frac{C_1\left(\left(2e^{\lambda L}\alpha\lambda L - 2\left(\left(\frac{-\alpha}{2} + \lambda L\right)\cos(\lambda L) + \frac{1}{2}\sin(\lambda L)\alpha\right)(\lambda L - \alpha)\right)e^{-\lambda L} + 2(\lambda L + \alpha)e^{\lambda L}\left(\left(\frac{\alpha}{2} + \lambda L\right)\cos(\lambda L) + \frac{1}{2}\sin(\lambda L)\alpha\right)\right)}{\left(\left((-3\lambda L\alpha + \alpha^2)\cos(\lambda L) + 2\left(\frac{\alpha}{2} + \lambda L\right)(\lambda L - \alpha)\sin(\lambda L)\right)e^{\lambda L} - \alpha(\sin(\lambda L)^2 + \cos(\lambda L)^2)(\lambda L + \alpha)\right)}$$

$$C_4 = \frac{C_1\left(\left(-2e^{\lambda L}\alpha^2 + \left(\left(\frac{-\alpha}{2} + \lambda L\right)\sin(\lambda L) - \frac{1}{2}\cos(\lambda L)\alpha\right)(\lambda L - \alpha)\right)e^{-\lambda L} + 2\left(\left(\frac{\alpha}{2} + \lambda L\right)\sin(\lambda L) - \frac{1}{2}\cos(\lambda L)\alpha\right)e^{\lambda L}(\lambda L + \alpha)\right)}{\left(\left((2\lambda^2L^2 - \lambda L\alpha - \alpha^2)\sin(\lambda L) - 3\cos(\lambda L)\left(\lambda L - \frac{\alpha}{3}\right)\alpha\right)e^{-\lambda L} - \alpha(\sin(\lambda L)^2 + \cos(\lambda L)^2)(\lambda L + \alpha)\right)} \quad (8)$$

It is obvious the constants can only be determined as a function of a constant multiplier (here C_1).A non-dimensional parameter $\alpha = KL/EI$is introduced to fully define the beam system in Fig. 1. The value of λ for a wide range of practical beam systems is given in Table 1 after solving equation (6). Apparently α = 0 corresponds to the simply-supported case and if $\alpha > 200$ support conditions converge to the fully-clamped case. The authors have attempted to provide a wide range of λ_n's corresponding to various values of α for practical conceptual engineering design purposes. The truncation at λ_7 gives a very accurate prediction of the displacement profile for an impulsively loaded beam. Though in most cases, truncating at n =3 gives a very good approximation. The Fourier series transform of most common pulses from time to frequency domain affirms this result [12]. However, readers interested in determining, analytically, higher modes can obtain themby equating the determinant of the positive definite matrix obtained by inputting the boundary conditions of equations (3)–(5) into equation (7) to zero. The reader can asymptotically compare the values and mode shapes to those of the fully-clamped and simply-supported conditions derived by Polyanin [13] given in equations (9) and (10), respectively.

$$\phi_n(x) = [(\sinh(\lambda_n L) - \sin(\lambda_n L))(\cosh(\lambda_n x) - \cos(\lambda_n x))] - [(\cosh(\lambda_n L) - \cos(\lambda_n L))(\sinh(\lambda_n x) - \sin(\lambda_n x))] \quad (9a)$$

$$\lambda_n = \frac{U_n}{L}, U_1 = 1.875, \quad U_2 = 4.694, \; U_n = \frac{\pi}{2}(2n-1) \text{ for } n \geq 3 \quad (9b)$$

$$\phi_n(x) = \sin(\lambda_n L) \quad (10a)$$

$$\lambda_n = \frac{U_1}{L} n = 1 \ldots\ldots n \quad (10b)$$

Table 1. λ_n corresponding various values of α

	α =0	α =1	α =2	α =3
λ_1	3.14	3.40	3.58	3.71
λ_2	6.28	6.43	6.55	6.65
λ_3	9.43	9.52	9.61	9.69
λ_4	12.57	12.64	12.71	12.78
λ_5	15.71	15.77	15.83	15.88
λ_6	18.85	18.90	18.95	19.00
λ_7	21.99	22.036	22.08	22.12
	α =5	**α =6**	**α =7**	**α =8**
λ_1	3.90	3.97	4.03	4.08
λ_2	6.81	6.87	6.93	6.98
λ_3	9.83	9.88	9.94	9.98
λ_4	12.89	12.94	12.99	13.03
λ_5	15.98	16.02	16.06	16.10
λ_6	19.08	19.12	19.16	19.19
λ_7	22.19	22.23	22.26	22.29
	α =9	**α =10**	**α =100**	**α =200**
λ_1	4.12	4.16	4.64	4.68
λ_2	7.03	7.07	7.71	7.78
λ_3	10.03	10.07	10.81	10.89
λ_4	13.07	13.17	13.89	14.01
λ_5	16.14	16.17	16.99	17.12
λ_6	19.22	19.26	20.09	20.24
λ_7	22.32	22.35	23.19	23.35

4. SOLUTION OF THE EQUATION OF MOTION

To solve the partial differential equation in equation (2), the solution is formulated as a series expansion as follows:

$$w_i(x,t) = a_{i\beta}(t)\phi_\beta(x) \tag{11}$$

Where $a_{i\beta}(t)$ are the generalised coordinates and $\phi_\beta(x)$ are the shape functions that satisfy boundary conditions (2) - (5). Summation convention is implied here.

The equation of motion (equation (2)) is re-written in the following form:

$$L_{ij}U_j + f_i = 0 \tag{12}$$

Using Galerkin's method to convert the partial differential equations to ordinary differential equation for easy computation we have:

$$\int_0^L w_\beta\left(L_{ij}U_{j\beta}\right)\mathrm{d}x + \int_0^L w_\beta f_i \mathrm{d}x = 0 \tag{13}$$

Where

$$w_\beta = \phi_\beta$$

For instance for α = 6, the corresponding shape function are shown in equations (14)

$$\phi_1(x) = e^{3.97x} + 52.76e^{-3.97x} + 122.56\sin(3.97x) - 53.76\cos(3.97x)$$

$$\phi_2(x) = e^{6.87x} - 964.99e^{-6.87x} - 3186.99\sin(6.87x) + 963.99\cos(6.87x)$$

$$\phi_3(x) = e^{9.88x} + 19562.32e^{-9.88x} + 84400.84\sin(9.88x) - 19563.32\cos(9.88x)$$

$$\phi_4(x) = e^{12.94x} - 4.15\times10^5e^{-12.94x} + 2.20\times10^6\sin(12.94x) + 4.15\times10^5\cos(12.94x)$$

$$\phi_5(x) = e^{16.02x} + 9.06\times10^6e^{-16.02x} + 5.76\times10^7\sin(16.02x) - 9.06\times10^6\cos(16.02x)$$

$$\phi_6(x) = e^{19.12x} + 2.01\times10^8e^{-19.12x} - 2.82\times10^7\sin(19.12x) + 2.01\times10^8\cos(19.12x)$$

$$\phi_7(x) = e^{22.22x} + 4.49\times10^9e^{-22.22x} + 3.9\times10^{10}\sin(22.22x) - 4.49\times10^9\cos(22.22x) \tag{14}$$

The corresponding shape functions from the modes 1 to 7 are derived as shown graphically in Fig. 2.

Therefore the solution, truncated at n=7 is

$$w(x,t) = a_1(t)\phi_1(x) + a_2(t)\phi_2(x) + a_3(t)\phi_3(x) + a_4(t)\phi_4(x) + a_5(t)\phi_5(x) + a_6(t)\phi_6(x) + a_7(t)\phi_7(x) \tag{15}$$

Equation (15) is inputted into the partial differential equations, PDE, of equation (2) and using Galerkin's method of weighted residuals the relevant ordinary differential equations, ODE's are obtained. It can be seen that an infinite degree-of-freedom system (infinite series) is developed in analysing the system. Truncating at say n =7 gives a 7 degree-of-freedom system which gives a high accuracy for must pulse loads [12]. A MATLAB code was developed to analyse the whole process presented in this work, which allows the user to determine the point of truncation depending on the time available for analysis and CPU speed. The input variables are α, K_θ, L, EI, A, ρ, $P(t)$.

5. EXAMPLE

Table 2 shows the equivalent geometric and mechanical properties of a practical blast wall (which can be referred to as a semi-rigidly supported beam) with boundary conditions depicting end connections to the top and bottom deck of a platform. The maximum displacement at the mid span of the blast wall, when subjected to a pulse load shown in Fig. 3, is obtained with the procedure presented in this paper.

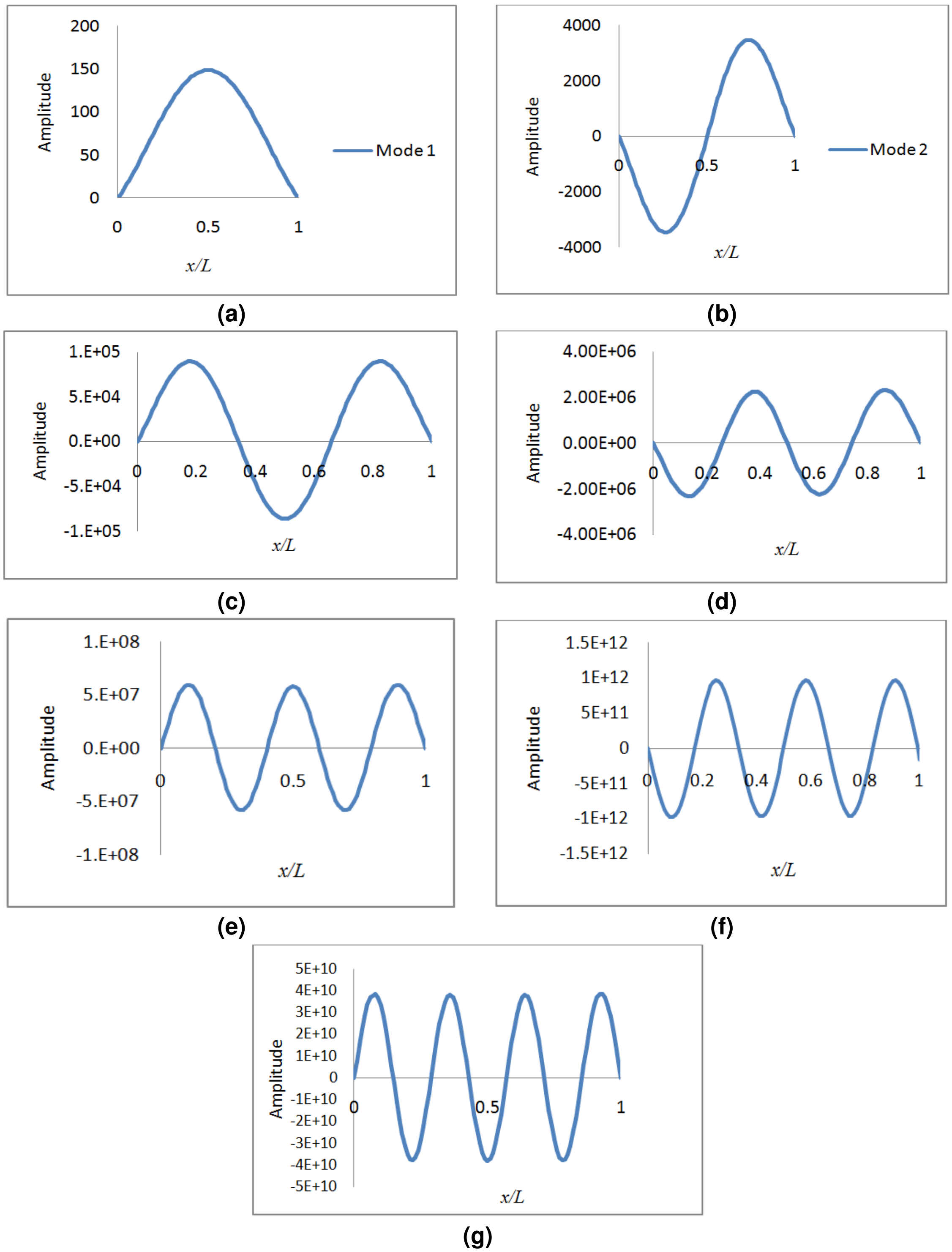

Fig. 2. Exact shape functions, φ_1 to φ_7. (a) Shape function, ϕ_1 (Mode 1) (b) Shape function, ϕ_2 (Mode 2) (c) Shape function, ϕ_3 (Mode 3) (d) Shape function, ϕ_4 (Mode 4) (e) Shape function, ϕ_5 (Mode 5) (f) Shape function, ϕ_6 (Mode 6) (g) Shape function, ϕ_7 (Mode 7)

Table 2. Shows properties of a semi-rigidly supported beam

Beam properties	Values	Dimension
b, width	0.1	m
d, dept	0.01	m
L, length of member	1	m
A, cross section area	0.001	m^2
E, young modulus of steel	2.00E+11	N/m^2
I, section modulus	8.33E-09	N/mm^2
K_θ, support spring stiffness	9990	Nm/rad
ρ, density	7850	Kg/m^3

An applied triangular pulse load similar to the profile generated by a high explosive detonation is shown in Fig. 3 and represented analytically in Equation 16.

$$P(x,t) = \begin{pmatrix} 1600\left(1 - \dfrac{t}{0.08}\right) & t \le 0.08s \\ 0 & t > 0.08s \end{pmatrix} \frac{N}{mm^2} \tag{16}$$

The closed form solution from Polyanin which only addresses traditional support condition is shown in Equations (17) – (19)

$$w(x,t) = \frac{\partial}{\partial t}\int_0^L f(\xi)G(x,\xi,t)\,\mathrm{d}\xi + \int_0^L g(\xi)G(x,\xi,t)\,\mathrm{d}\xi + \int_0^t\int_0^L \phi(x,\tau)G(x,\xi,t-\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{17}$$

where the Green function, *G* (*x, ξ, t*),is as equation (17):

$$G(x,\xi,t) = \frac{1}{a}\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)}{\lambda_n^2\|\varphi_n\|^2}\sin(\lambda_n^2 at) \tag{18}$$

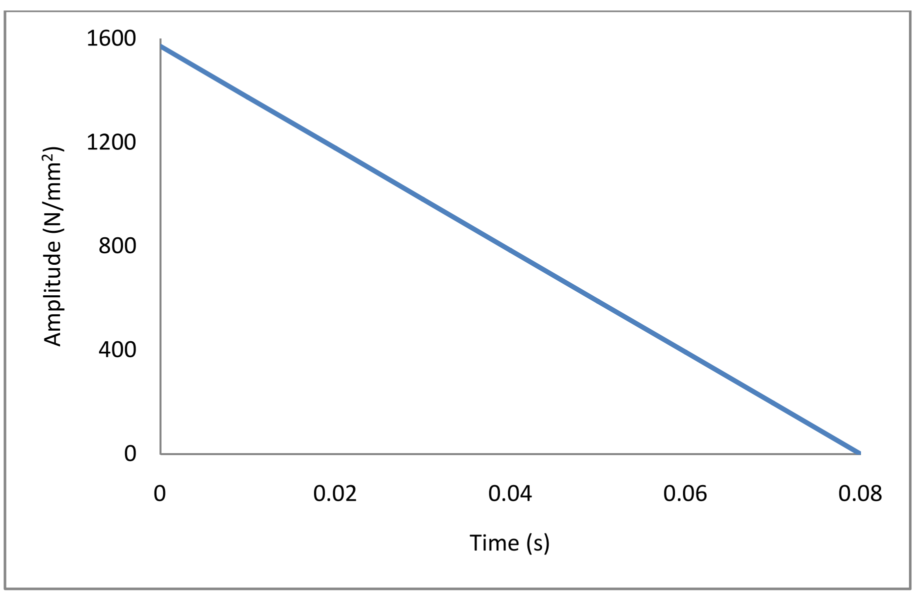

Fig. 3. Applied triangular pulse load

where the term $\|\phi_n\|^2$ is

$$\|\phi_n\|^2 = \int_0^L \phi_n^2(x)\mathrm{d}x = \frac{L}{4}\phi_n^2(L) + \frac{L}{4\lambda_n^4}\left[\phi_n''(L)\right]^2 - \frac{L}{2\lambda_n^4}\phi_n'(L)\phi_n'''(L) \tag{19}$$

The maximum displacement occurs at the mid span and is predicted by the presented procedure in this paper. This is compared with exact solution of Polyanin [13] and the correlation is very strong as shown in Fig. 4.

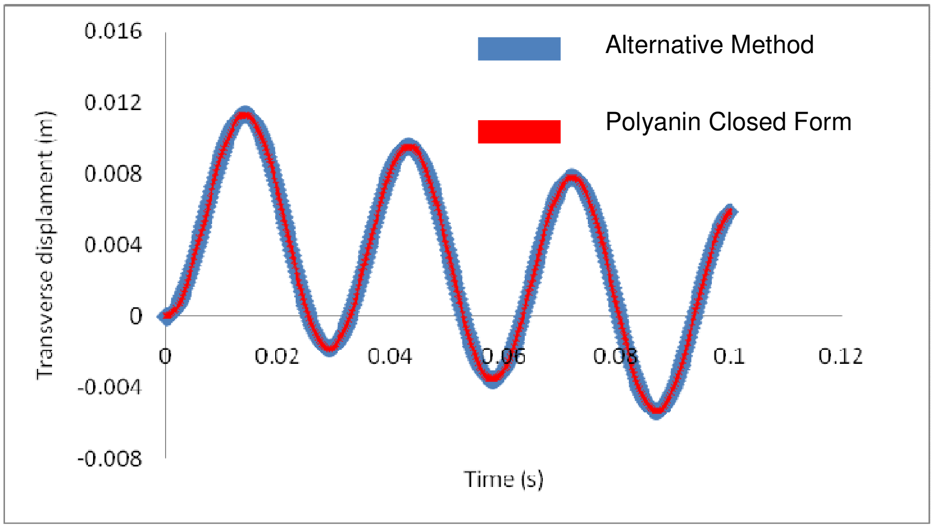

Fig. 4. Maximum displacements at mid span from presented method and Polyanin

6. CONCLUSION

This paper addresses the transverse displacement of practical beam systems with semi-rigid rotational spring boundary conditions. This has not been addressed by researchers who have used complex and modern mathematical algorithms to analyse dynamically loaded beams. They have concentrated on the traditional simplistic support conditions of perfect hinge and perfect clamps.

The pseudo-design too land procedure presented in this work serves as a tool for conceptual engineering design in estimating the behaviour of dynamically loaded beams with practical supports with an estimated rotational property at hand. Engineers will have an option of quickly assessing the displacement profile and subsequent location of maximum moments (the designer might want to have maximum moments at supports or within the beam) and stress in the beam system. The values presented in the analysis table converge to a fully clamped solution for $\alpha > 200$. Thus providing a guide for estimation for a simplified idealisation.

When compared with the works done by Hsu, Schleyer and Langdon [1,2] on blast walls, the benefits of the procedure becomes more lucid and indispensable. This procedure provides a more accurate and quick assessment of a blast wall with semi rigid connections, though with simpler support parameters. The numerical model presented by Schleyer and his team only considers overall modes of the systems and does not involve higher modes.

The results presented are compared with the results of the full deflection predicted by using the modified Polyanin's closed form approach [13] and a perfect correlation was achieved.

COMPETING INTERESTS

Authors declare that there are no competing interests.

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