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A Sufficient Convexity Condition for Parametric Bézier Surface over Rectangle

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Abstract

Surface convexity is a key issue in computer aided geometric design, which is widely applied in geometric modeling field, such as physical models, industrial design, automatic manufacturing, etc. In this paper, a sufficient convexity condition of the parametric Bézier surface over rectangles is proposed, which is firstly considered as a sufficient convexity condition for the Bézier control grid. The condition is proved by De Casteljau surface subdivision arithmetic, in which the recursive expressions elaborate that the control grid eventually converges to the surface. At last, two examples for the modeling of interpolation-type surface are discussed, one of which is a general surface and the other is a degenerate surface.

Keywords

Convexity Condition, Bézier Surface, De Casteljau Arithmetic, Interpolation-Type Surface

1. Introduction

Surface construction or design is one of the most concerned issues in computer aided geometric design (CAGD) [1]. Parametric surface provides flexible control over surface parameters and shapes, which is widely used in surface modeling fields [2] [3] [4]. The Bézier curve and surface expressions were proposed and developed by P. Bézier and P. De Casteljau in the middle of 20th century [5]. Its parametric representation is free of dependence on coordinates, which brings great convenience to designers. It has become the main tool for describing shape information in surface engineering technology [6].

In the study of surface design, convexity is an important surface property in the process of surface constructions [7] [8] [9]. In the field of material or mechanical engineering, surface convexity has significant application in free surface modeling, such as vehicle body surface design and testing, physical geometry construction [7], etc. For example, in the metal plasticity theory, the yield surface in stress space is required to obey Drucker postulate, which means it should be convex [10]. For Bézier surface, many scholars have given convexity conditions about various surface forms, including the parametric or nonparametric Bézier surface patches. Several convexity conditions of the surfaces over triangles have been established, especially the non-parametric triangular Bézier surfaces [11] [12] [13]. On the other hand, the convexity of tensor-product Bézier surfaces over rectangles is relatively more difficult to guarantee [14]. There are two main types of conditions. One is based on the real function, and it needs to ensure the semi-definiteness of the Hessian matrix corresponding to the function. M. Floater [15], W. Dahmen [16], Y. Ding [17] proposed convexity conditions based on this theory. For example, the necessary and sufficient conditions for a $m \times n$ Bézier surface (the corresponding control vertex is denoted as $p_{i,j}$) proposed by *M*. Floater to be convex are:

$$\begin{cases} a_{i,j} \ge 0, i = 0, \cdots, m-2;\ j = 0, \cdots, n, \\ c_{i,j} \ge 0, i = 0, \cdots, m;\ j = 0, \cdots, n-2, \\ a_{i,j+l+s} c_{i+k+r,j} \ge b_{i+k,j+l}^2, i = 0, \cdots, m-2;\ j = 0, \cdots, n-2 \end{cases} \quad (1)$$

where

$$\begin{cases} a_{i,j} = m(m-1)\left(p_{i,j} - 2p_{i+1,j} + p_{i+2,j}\right) \\ b_{i,j} = mn\left(p_{i,j} - p_{i+1,j} - p_{i,j+1} + p_{i+1,j+1}\right) \\ c_{i,j} = n(n-1)\left(p_{i,j} - 2p_{i,j+1} + p_{i,j+2}\right) \end{cases}$$

The other is based on the vector function, which is achieved by ensuring that the full curvature of every point on the surface is greater than zero. G. Koras [18] derived necessary and sufficient convexity conditions based on B-spline surface. J. Yao [19] proposed simplified convexity conditions and discussed the relationship between surface convexity and control grid convexity. Most of the proposed conditions are inconvenient or limitative for applying.

In this paper, based on a convexity condition of Bézier grid proposed by J. Yao [19], we prove that it happens to be a sufficient convexity condition of the Bézier surface. The subdivision of the Bézier surface is used to explain how the control grid converges to the surface. The second section shows some preliminary knowledge. The third section gives proof of the convexity condition, and the fourth section presents two application examples for surface modeling.

2. Theoretic Basis

2.1. Bézier Curves and Surfaces

The parametric Bézier curve and surface can be expressed as Equation (2) and Equation (3) respectively,

$$\boldsymbol{p}(t) = \sum_{i=0}^{n} \boldsymbol{b}_i B_{i,n}(t), 0 \le t \le 1 \quad (2)$$

$$\boldsymbol{r}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\boldsymbol{b}_{i,j}B_{i,m}(u)B_{j,n}(v),\ 0\le u,v\le 1 \tag{3}$$

where $B_{i,m}(t)=C_m^i t^i(1-t)^{m-i}, i=0,1,\cdots,m, j=0,1,\cdots,n$, $C_m^i=\dfrac{m!}{i!(m-i)!}$. t, u and v are inner variables, and $\boldsymbol{b}_i$ are vertex vectors.

2.2. De Casteljau Subdivision Arithmetic [20]

De Casteljau subdivision algorithm is constructed from a control polygon or grid, inserting new vertices according to certain subdivision rules, and connecting these new vertices to obtain new control polygons or control grids. The new control polygon or grid obtained from once subdivision process is used as the initial control geometry, and repeat the subdivision process above. Finally, the control polygons or grids generated by recursive subdivision converge to the curve or surface corresponding to the initial control polygon or grid.

For Bézier curves, the arithmetic is expressed as,

$$\boldsymbol{b}_j^k=\begin{cases}\boldsymbol{b}_j, k=0\\(1-t)\boldsymbol{b}_j^{k-1}+t\boldsymbol{b}_{j+1}^{k-1}, k=1,2,\cdots,n\end{cases},\ j=0,1,\cdots,n-k \tag{4}$$

where $0\le t\le 1$ and $\boldsymbol{b}_j^k$ are middle control points. Take $n=3$, $t=0.5$ as an example, two new control polygons are expressed as $\{\boldsymbol{b}_0^k\}$ and $\{\boldsymbol{b}_k^{3-k}\}$ respectively and the subdivision process can be shown as Figure 1.

Similarly, for Bézier surface, the corresponding subdivision (dichotomy) arithmetic is expressed as,

$$\boldsymbol{b}_{i,j}^k=\begin{cases}\boldsymbol{b}_{i,j}, k=0\\(1-t)\boldsymbol{b}_{i,j}^{k-1}+t\boldsymbol{b}_{i,j+1}^{k-1}, k=1,2,\cdots,n\end{cases},\ i=0,1,\cdots,n;\ j=0,1,\cdots,n-k \tag{5}$$

where $0\le t\le 1$ and $\boldsymbol{b}_{i,j}^k$ are middle control points. Take $m=n=3$, $t=0.5$ as an example, two new control grids are expressed as $\{\boldsymbol{b}_{i,0}^k\}$ and $\{\boldsymbol{b}_{i,3-k}^k\}$ respectively and the subdivision process can be shown as Figure 2.

2.3. Convexity Condition for Bézier Grid

Lemma [19] For a $m\times n$ Bézier grid, if the control vertices satisfy the following conditions:

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{2,0}\boldsymbol{b}_{s,t}\right)\ge 0(\le 0)\\\left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{0,2}\boldsymbol{b}_{\alpha,\beta}\right)\ge 0(\le 0)\end{cases} \tag{6}$$

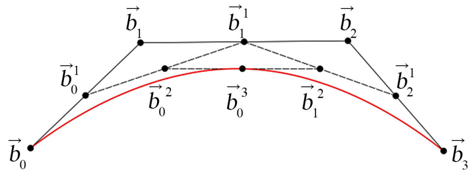

Figure 1. A Third-order Bézier curve and its middle control vertices after once subdivision by De Casteljau arithmetic.

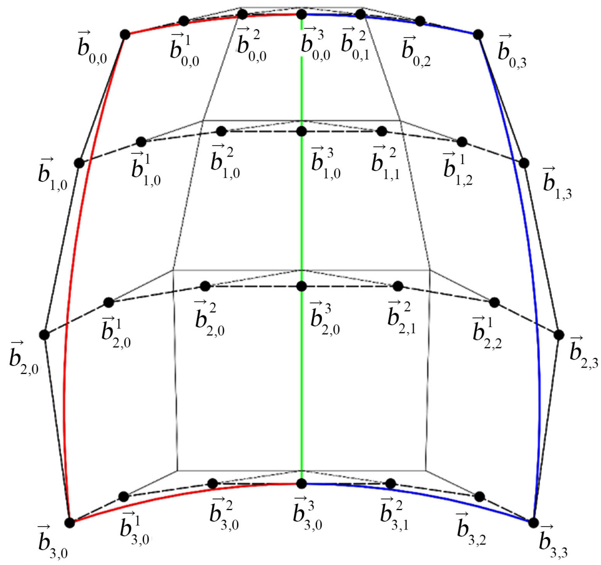

Figure 2. A 3 × 3 Bézier surface and its middle control vertices after once equant subdivision along v direction by De Casteljau arithmetic.

$$i = 0,1,\cdots,m-1;\ j,t = 0,1,\cdots,n;\ p,\alpha = 0,1,\cdots,m;$$
$$q = 0,1,\cdots,n-1;\ s = 0,1,\cdots,m-2;\ \beta = 0,1,\cdots,n-2$$

then the Bézier grid is downward (upward) convex.

In this expression, $\Delta^{1,0}\boldsymbol{b}_{i,j}$, $\Delta^{0,1}\boldsymbol{b}_{i,j}$ and $\Delta^{2,0}\boldsymbol{b}_{i,j}$, $\Delta^{0,2}\boldsymbol{b}_{i,j}$ represent the first-order difference and second-order difference of the control grid, respectively. It reflects the positive and negative consistency of all possible mixed products with two first-order differences and one second-order difference. In fact, the necessary and sufficient conditions of the convex Bézier surface proposed by Floater [15] and Koras [18] include three inequalities, and Equation (6) essentially corresponds to the first two inequalities in the necessary and sufficient conditions (Equation (1)). The following is Yao's proof of the lemma.

Definition [19] For a $m\times n$ Bézier grid, if the control vertices satisfy the following conditions:

$$\begin{cases} \boldsymbol{l}_{i,j}\cdot\left(\boldsymbol{b}_{i+k_1,j}-\boldsymbol{b}_{i,j}\right)\ge 0(\le 0) \\ \boldsymbol{l}_{i,j}\cdot\left(\boldsymbol{b}_{i,j+k_2}-\boldsymbol{b}_{i,j}\right)\ge 0(\le 0) \end{cases} \tag{7}$$

where

$$\boldsymbol{l}_{i,j} = \Delta^{1,0}\boldsymbol{b}_{i,j+k_2}\times\Delta^{0,1}\boldsymbol{b}_{i+k_1,j},$$
$$i = 0,1,\cdots,m;\ j = 0,1,\cdots,n;\ k_1 = 1,2,\cdots,m-i;\ k_2 = 1,2,\cdots,n-j,$$

then the Bézier grid is downward (upward) convex.

Proof. [19] If the vertices of the Bézier grid satisfy the Equation (6), it can be deduced that

$$\begin{cases} \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{2,0}\boldsymbol{b}_{i+p,j}\right)\ge 0(\le 0) \\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{0,2}\boldsymbol{b}_{i,j+q}\right)\ge 0(\le 0) \end{cases}$$

$$i = 0,1,\cdots,m-i-2;\ j = 0,1,\cdots,n-j-2.$$

Then,

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{1,0}\boldsymbol{b}_{i+p+1,j}-\Delta^{1,0}\boldsymbol{b}_{i+p,j}\right)\geq 0(\leq 0)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{0,1}\boldsymbol{b}_{i,j+q+1}-\Delta^{0,1}\boldsymbol{b}_{i,j+q}\right)\geq 0(\leq 0)\end{cases}$$

Then,

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\sum_{r=1}^{k_1}\sum_{p=0}^{r-2}\left(\Delta^{1,0}\boldsymbol{b}_{i+p+1,j}-\Delta^{1,0}\boldsymbol{b}_{i+p,j}\right)\right)\geq 0(\leq 0)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\sum_{r=1}^{k_2}\sum_{q=0}^{r-2}\left(\Delta^{0,1}\boldsymbol{b}_{i,j+q+1}-\Delta^{0,1}\boldsymbol{b}_{i,j+q}\right)\right)\geq 0(\leq 0)\end{cases}$$

$$k_1 = 1,2,\cdots,m-i;\ k_2 = 1,2,\cdots,n-j.$$

Then,

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\boldsymbol{b}_{i+k_1,j}-\boldsymbol{b}_{i,j}\right)\geq 0(\leq 0)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\boldsymbol{b}_{k,j+k_2}-\boldsymbol{b}_{i,j}\right)\geq 0(\leq 0)\end{cases}$$

and it leads to Equation (7).

3. Proof of the Convexity Conditions for Bézier Surface

Generally, the Bézier grid convexity does not necessarily lead to the Bézier surface convexity and vice versa. W. Dahmen [16] proposed that if the Bézier control grid is a piecewise bilinear surface that interpolates control vertices, the convex control grid can lead to a convex surface, which is also called a translational surface. In fact, the control grid corresponding to translational surface is not only bilinear but also linear in each patch [15], *i.e.*, the control grid is composed of parallelograms [16]. Yao showed the necessary and sufficient conditions for a Bézier control grid to be convex:

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{2,0}\boldsymbol{b}_{s,t}\right)\geq 0(\leq 0)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{0,2}\boldsymbol{b}_{\alpha,\beta}\right)\geq 0(\leq 0)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j},\Delta^{0,1}\boldsymbol{b}_{p,q},\Delta^{0,2}\boldsymbol{b}_{\gamma,\delta}\right)= 0\end{cases}$$

This condition is a further limitation on Equation (6), so that the shape of the control mesh is limited to parallelograms, which rules out the vast majority of surfaces [15].

Base on the theory in section 2.2, we propose a new idea that if the convex grid which is defined by a certain convexity condition can keep the convexity in the process of subdivision, the new convex grids would converge to the convex points forming the surface. The convex condition for Bézier grid proposed by J. Yao [19] is a sufficient condition and it can be proved keeping convex in the process of surface subdivision by De Casteljau arithmetic. In this proof, take $m = n$, $t = 0.5$ and the sub-grid $\left\{\boldsymbol{b}_{i,0}^{k}\right\}$ as an example, give once subdivision process and derive related differential expressions.

Definition.

$$\begin{cases} \boldsymbol{b}_{i+1,0}^{k} - \boldsymbol{b}_{i,0}^{k} \triangleq \Delta^{1,0}\boldsymbol{b}_{i,0}^{k} \\ \boldsymbol{b}_{i,0}^{k+1} - \boldsymbol{b}_{i,0}^{k} \triangleq \Delta^{0,1}\boldsymbol{b}_{i,0}^{k} \\ \Delta^{1,0}\boldsymbol{b}_{i+1,0}^{k} - \Delta^{1,0}\boldsymbol{b}_{i,0}^{k} \triangleq \Delta^{2,0}\boldsymbol{b}_{i,0}^{k} \\ \Delta^{0,1}\boldsymbol{b}_{i,0}^{k+1} - \Delta^{0,1}\boldsymbol{b}_{i,0}^{k} \triangleq \Delta^{0,2}\boldsymbol{b}_{i,0}^{k} \end{cases} \tag{8}$$

Proof.

1) When $k=0$,

$$\Delta^{1,0}\boldsymbol{b}_{i,0}^{k} = \boldsymbol{b}_{i+1,0}^{0} - \boldsymbol{b}_{i,0}^{0} = \Delta^{1,0}\boldsymbol{b}_{i,0}$$

$$\Delta^{0,1}\boldsymbol{b}_{i,0}^{k} = \boldsymbol{b}_{i,0}^{1} - \boldsymbol{b}_{i,0}^{0} = 1/2\left(\boldsymbol{b}_{i,0} + \boldsymbol{b}_{i,1}\right) - \boldsymbol{b}_{i,0} = 1/2\,\Delta^{0,1}\boldsymbol{b}_{i,0}$$

$$\Delta^{2,0}\boldsymbol{b}_{i,0}^{k} = \Delta^{1,0}\boldsymbol{b}_{i+1,0} - \Delta^{1,0}\boldsymbol{b}_{i,0} = \Delta^{2,0}\boldsymbol{b}_{i,0}$$

$$\Delta^{0,2}\boldsymbol{b}_{i,0}^{k} = \Delta^{0,1}\boldsymbol{b}_{i,0}^{1} - \Delta^{0,1}\boldsymbol{b}_{i,0}^{0} = 1/2\left(\Delta^{0,1}\boldsymbol{b}_{i,0}^{0} + \Delta^{0,1}\boldsymbol{b}_{i,1}^{0}\right) - \Delta^{0,1}\boldsymbol{b}_{i,0}^{0} = 1/4\,\Delta^{0,2}\boldsymbol{b}_{i,0}$$

2) when $k=1,2,\cdots,n$,

$$\begin{aligned}\Delta^{1,0}\boldsymbol{b}_{i,0}^{k} &= 1/2\left(\boldsymbol{b}_{i+1,0}^{k-1} + \boldsymbol{b}_{i+1,1}^{k-1}\right) - 1/2\left(\boldsymbol{b}_{i,0}^{k-1} + \boldsymbol{b}_{i,1}^{k-1}\right) \\ &= 1/2\left(\Delta^{1,0}\boldsymbol{b}_{i,0}^{k-1} + \Delta^{1,0}\boldsymbol{b}_{i,1}^{k-1}\right) \\ &= \sum_{k'=0}^{k} \frac{C_k^{k'}}{2^k}\Delta^{1,0}\boldsymbol{b}_{i,k'}\end{aligned}$$

$$\begin{aligned}\Delta^{0,1}\boldsymbol{b}_{i,0}^{k} &= \frac{1}{2}\left(\boldsymbol{b}_{i,0}^{k} + \boldsymbol{b}_{i,1}^{k}\right) - 1/2\left(\boldsymbol{b}_{i,0}^{k-1} + \boldsymbol{b}_{i,1}^{k-1}\right) \\ &= 1/2\left(\Delta^{0,1}\boldsymbol{b}_{i,0}^{k-1} + \Delta^{0,1}\boldsymbol{b}_{i,1}^{k-1}\right) \\ &= \sum_{k'=0}^{k} \frac{C_k^{k'}}{2^{k+1}}\Delta^{0,1}\boldsymbol{b}_{i,k'}\end{aligned}$$

$$\begin{aligned}\Delta^{2,0}\boldsymbol{b}_{i,0}^{k} &= 1/2\left(\Delta^{1,0}\boldsymbol{b}_{i+1,0}^{k-1} + \Delta^{1,0}\boldsymbol{b}_{i+1,1}^{k-1}\right) - 1/2\left(\Delta^{1,0}\boldsymbol{b}_{i,0}^{k-1} + \Delta^{1,0}\boldsymbol{b}_{i,1}^{k-1}\right) \\ &= 1/2\left(\Delta^{2,0}\boldsymbol{b}_{i,0}^{k-1} + \Delta^{2,0}\boldsymbol{b}_{i,1}^{k-1}\right) \\ &= \sum_{k'=0}^{k} \frac{C_k^{k'}}{2^k}\Delta^{2,0}\boldsymbol{b}_{i,k'}\end{aligned}$$

$$\begin{aligned}\Delta^{0,2}\boldsymbol{b}_{i,0}^{k} &= \frac{1}{2}\left(\Delta^{0,1}\boldsymbol{b}_{i,0}^{k} + \Delta^{1,0}\boldsymbol{b}_{i,1}^{k}\right) - 1/2\left(\Delta^{0,1}\boldsymbol{b}_{i,0}^{k-1} + \Delta^{0,1}\boldsymbol{b}_{i,1}^{k-1}\right) \\ &= 1/2\left(\Delta^{0,2}\boldsymbol{b}_{i,0}^{k-1} + \Delta^{0,2}\boldsymbol{b}_{i,1}^{k-1}\right) \\ &= \sum_{k'=0}^{k} \frac{C_k^{k'}}{2^{k+2}}\Delta^{0,2}\boldsymbol{b}_{i,k'}\end{aligned}$$

where $C_k^{k'} = \dfrac{k!}{k'!\left(k-k'\right)!}$. In summary,

$$\Delta^{1,0}\boldsymbol{b}_{i,0}^{k} = \sum_{k'=0}^{k} \frac{C_k^{k'}}{2^k}\Delta^{1,0}\boldsymbol{b}_{i,k'},\ k=0,1,\cdots,n;\ i=0,1,\cdots,n-1 \tag{9}$$

$$\Delta^{0,1}\boldsymbol{b}_{i,0}^{k} = \sum_{k'=0}^{k} \frac{C_k^{k'}}{2^{k+1}}\Delta^{0,1}\boldsymbol{b}_{i,k'},\ k=0,1,\cdots,n-1;\ i=0,1,\cdots,n \tag{10}$$

$$\Delta^{2,0}\boldsymbol{b}_{i,0}^{k}=\sum_{k'=0}^{k}\frac{C_{k}^{k'}}{2^{k}}\Delta^{2,0}\boldsymbol{b}_{i,k'},\ k=0,1,\cdots,n;\ i=0,1,\cdots,n-2 \tag{11}$$

$$\Delta^{0,2}\boldsymbol{b}_{i,0}^{k}=\sum_{k'=0}^{k}\frac{C_{k}^{k'}}{2^{k+2}}\Delta^{0,2}\boldsymbol{b}_{i,k'},\ k=0,1,\cdots,n-2;\ i=0,1,\cdots,n \tag{12}$$

As a result,

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i_1,0}^{k_1},\Delta^{0,1}\boldsymbol{b}_{i_2,0}^{k_2},\Delta^{2,0}\boldsymbol{b}_{i_3,0}^{k_3}\right)=\sum\limits_{k_1',k_2',k_3'}\varphi_1\left(\Delta^{1,0}\boldsymbol{b}_{i_1,k_1'},\Delta^{0,1}\boldsymbol{b}_{i_2,k_2'},\Delta^{2,0}\boldsymbol{b}_{i_3,k_3'}\right)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i_1,0}^{k_1},\Delta^{0,1}\boldsymbol{b}_{i_2,0}^{k_2},\Delta^{0,2}\boldsymbol{b}_{i_4,0}^{k_4}\right)=\sum\limits_{k_1',k_2',k_4'}\varphi_2\left(\Delta^{1,0}\boldsymbol{b}_{i_1,k_1'},\Delta^{0,1}\boldsymbol{b}_{i_2,k_2'},\Delta^{0,2}\boldsymbol{b}_{i_4,k_4'}\right)\end{cases} \tag{13}$$

$$i_2,i_4,k_1,k_3=0,1,\cdots,n;\ i_1,k_2=0,1,\cdots,n-1;\ i_3,k_4=0,1,\cdots,n-2$$

where $\varphi_1=C_{k_1}^{k_1'}C_{k_2}^{k_2'}C_{k_3}^{k_3'}/2^{k_1+k_2+k_3+1}$, $\varphi_2=C_{k_1}^{k_1'}C_{k_2}^{k_2'}C_{k_4}^{k_4'}/2^{k_1+k_2+k_4+2}$ and all of them are positive.

Consequently, combined with Equation (6), it concludes as,

$$\begin{cases}\left(\Delta^{1,0}\boldsymbol{b}_{i_1,0}^{k_1},\Delta^{0,1}\boldsymbol{b}_{i_2,0}^{k_2},\Delta^{2,0}\boldsymbol{b}_{i_3,0}^{k_3}\right)\geq 0(\leq 0)\\ \left(\Delta^{1,0}\boldsymbol{b}_{i_1,0}^{k_1},\Delta^{0,1}\boldsymbol{b}_{i_2,0}^{k_2},\Delta^{0,2}\boldsymbol{b}_{i_4,0}^{k_4}\right)\geq 0(\leq 0)\end{cases} \tag{14}$$

It is proved that the sub-grid with control vertices of $\left\{\boldsymbol{b}_{i,0}^{k}\right\}$ is convex, so in any subdivision process, the new control sub-grids obtained are convex. This proof shows that for a Béziergrid that satisfies the convexity condition (Equation (6)), the corresponding surface is also convex.

4. Applications

The new sufficient convexity condition can be used to determine the convexity of a given Bézier patch, which is available for the modeling of interpolation-type surface. For a rectangular patch, four corner vertices are interpolation nodes, and both coordinates and corresponding normal can contribute to construct the control grid. Specifically, the coordinate of interpolation node represents the position data of the test point, and the cross-product of two vectors formed by the interpolation node and two other nodes connected thereto represents the normal vector at the interpolation point.

4.1. Example 1

For a 2 × 2 Bézier patch (see **Figure 3**), the external control vertices have been given by experimental data, and only the internal vertex $\boldsymbol{b}_{1,1}$ is unknown. Specifically, four corner vertices $\boldsymbol{b}_{0,0}$, $\boldsymbol{b}_{0,2}$, $\boldsymbol{b}_{2,0}$ and $\boldsymbol{b}_{2,2}$ represent the experimental points and they are on the surface; $\boldsymbol{b}_{0,1}$, $\boldsymbol{b}_{1,0}$, $\boldsymbol{b}_{2,1}$ and $\boldsymbol{b}_{1,2}$ are determined by the normal data of these corner vertices, and they are not on the surface. If these mixed products unrelated to $\boldsymbol{b}_{1,1}$ have satisfied the convexity condition, $\boldsymbol{b}_{1,1}$ is the only point to determine the shape of the surface patch. Therefore, the difference terms associated with $\boldsymbol{b}_{1,1}$ need to be substituted into the convexity formulas to determine the values of the mixed product as shown below:

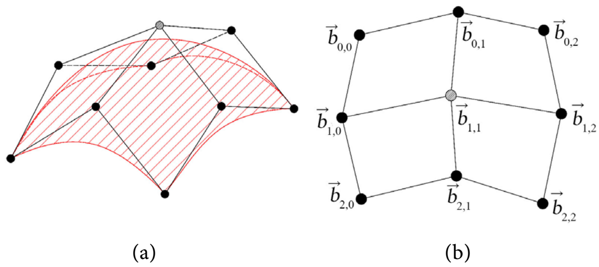

Figure 3. Diagrams for (a) 2 × 2 Bézier surface patch with control grid, and (b) 2 × 2 Bézier control grid.

$$\begin{cases} \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{2,0}\boldsymbol{b}_{0,1}\right) \\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{0,2}\boldsymbol{b}_{1,0}\right) \\ \left(\Delta^{1,0}\boldsymbol{b}_{i,1}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{2,0}\boldsymbol{b}_{s,t}\right) \\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{1,q}, \Delta^{0,2}\boldsymbol{b}_{\alpha,\beta}\right) \end{cases} \tag{15}$$

$$i,q = 0,1;\ j,t,p,\alpha = 0,1,2;\ s,\beta = 0$$

For example, the control grid of a Bézier patch is represented as

$$\boldsymbol{b}_{i,j} = \begin{pmatrix} (0,0,0) & (0,1,1) & (0,2,0) \\ (1,0,1) & \boldsymbol{b}_{1,1} & (1,2,1) \\ (2,0,0) & (2,1,1) & (2,2,0) \end{pmatrix},$$

where $\boldsymbol{b}_{1,1} = (x,y,z)$.

Then,

$$\Delta^{1,0}\boldsymbol{b}_{i,j} = \begin{bmatrix} (1,0,1) & (x,y-1,z-1) & (1,0,1) \\ (1,0,-1) & (2-x,1-y,1-z) & (1,0,-1) \end{bmatrix}$$

$$\Delta^{0,1}\boldsymbol{b}_{i,j} = \begin{bmatrix} (0,1,1) & (0,1,-1) \\ (x-1,y,z-1) & (1-x,2-y,1-z) \\ (0,1,1) & (0,1,-1) \end{bmatrix}$$

$$\Delta^{2,0}\boldsymbol{b}_{i,j} = \begin{bmatrix} (0,0,-2) & (2-2x,2-2y,2-2z) & (0,0,-2) \end{bmatrix}$$

$$\Delta^{0,2}\boldsymbol{b}_{i,j} = \begin{bmatrix} (0,0-2) \\ (2-2x,2-2y,2-2z) \\ (0,0,-2) \end{bmatrix}$$

Suppose there are two alternatives for $\boldsymbol{b}_{1,1}$, one is $(1,1,2)$, and the other is $(1,1,0)$. For $\boldsymbol{b}_{i,j} = (1,1,2)$,

$$\begin{cases} \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{2,0}\boldsymbol{b}_{s,t}\right) = -2 \le 0 \\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{0,2}\boldsymbol{b}_{\alpha,\beta}\right) = -2 \le 0 \end{cases}$$

So, it satisfies the convexity condition. For $\boldsymbol{b}_{i,j} = (1,1,0)$,

$$\begin{cases} \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{2,0}\boldsymbol{b}_{s,t}\right) = \{-2,2\} \\ \left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{0,2}\boldsymbol{b}_{\alpha,\beta}\right) = \{-2,2\} \end{cases}$$

Therefore, it does not satisfy the convexity condition.

4.2. Example 2

In some cases, a degenerate surface may be used to construct a surface with poles or sharp points, such as quasi-ellipsoid or quasi-cone, which causes the vertices of Bézier patch on one edge converging to a point. In this case, the number of interpolation nodes becomes three. As shown in Figure 4, a 3 × 3 planar Bézier-grid is constructed, of which the control grid is composed of plane polygons, *i.e.*, first row of triangles, second row of trapezoids and third row of rectangles.

Denote $\mathbf{AD}=\boldsymbol{h}_1$, $\mathbf{AF}=\boldsymbol{h}_2$, $\mathbf{DB}=\mathbf{FC}=\boldsymbol{v}$, $\mathbf{BE}=\boldsymbol{h}_3$ and $\mathbf{EC}=\boldsymbol{h}_4$, take $\mathbf{A}_1\mathbf{D}=a_1\boldsymbol{h}_1$, $\mathbf{A}_2\mathbf{F}=a_1\boldsymbol{h}_2$, $\mathbf{DB}_2=\mathbf{FC}_2=a_2\boldsymbol{v}$, $\mathbf{B}_1\mathbf{E}=a_3\boldsymbol{h}_3$ and $\mathbf{EC}_1=a_3\boldsymbol{h}_4$, where $0\le a_i\le 1$.

Denote $1-ja_i\triangleq a_i^j$. Obviously, $a_i,a_i^1\ge 0$. Then, the matrixes composed of the differential vectors can be expressed as:

$$\Delta^{1,0}\boldsymbol{b}_{i,j}=\begin{bmatrix} a_1^1\boldsymbol{h}_1 & a_1^1\left(\boldsymbol{h}_1+a_3^1\boldsymbol{h}_3\right) & a_1^1\left(\boldsymbol{h}_2-a_4^1\boldsymbol{h}_4\right) & a_1^1\boldsymbol{h}_2 \\ a_2\boldsymbol{v}+a_1\boldsymbol{h}_1 & a_2\boldsymbol{v}+a_1\left(\boldsymbol{h}_1+a_3^1\boldsymbol{h}_3\right) & a_2\boldsymbol{v}+a_1\left(\boldsymbol{h}_2-a_4^1\boldsymbol{h}_4\right) & a_2\boldsymbol{v}+a_1\boldsymbol{h}_2 \\ a_2^1\boldsymbol{v} & a_2^1\boldsymbol{v} & a_2^1\boldsymbol{v} & a_2^1\boldsymbol{v} \end{bmatrix}$$

$$\Delta^{0,1}\boldsymbol{b}_{i,j}=\begin{bmatrix} 0 & 0 & 0 \\ a_1^1a_3^1\boldsymbol{h}_3 & a_1^1\left(a_3\boldsymbol{h}_3+a_4\boldsymbol{h}_4\right) & a_1^1a_4^1\boldsymbol{h}_4 \\ a_3^1\boldsymbol{h}_3 & a_3\boldsymbol{h}_3+a_4\boldsymbol{h}_4 & a_4^1\boldsymbol{h}_4 \\ a_3^1\boldsymbol{h}_3 & a_3\boldsymbol{h}_3+a_4\boldsymbol{h}_4 & a_4^1\boldsymbol{h}_4 \end{bmatrix}$$

$$\Delta^{2,0}\boldsymbol{b}_{i,j}=\begin{bmatrix} a_2\boldsymbol{v}-a_1^2\boldsymbol{h}_1 & a_2\boldsymbol{v}-a_1^2\left(\boldsymbol{h}_1+a_3^1\boldsymbol{h}_3\right) & a_2\boldsymbol{v}-a_1^2\left(\boldsymbol{h}_2-a_4^1\boldsymbol{h}_4\right) & a_2\boldsymbol{v}-a_1^2\boldsymbol{h}_2 \\ a_2^2\boldsymbol{v}-a_1\boldsymbol{h}_1 & a_2^2\boldsymbol{v}-a_1\left(\boldsymbol{h}_1+a_3^1\boldsymbol{h}_3\right) & a_2^2\boldsymbol{v}-a_1\left(\boldsymbol{h}_2-a_4^1\boldsymbol{h}_4\right) & a_2^2\boldsymbol{v}-a_1\boldsymbol{h}_2 \end{bmatrix}$$

$$\Delta^{0,2}\boldsymbol{b}_{i,j}=\begin{bmatrix} 0 & 0 \\ a_1^1\left(a_4\boldsymbol{h}_4-a_3^2\boldsymbol{h}_3\right) & a_1^1\left(a_4^2\boldsymbol{h}_4-a_3\boldsymbol{h}_3\right) \\ a_4\boldsymbol{h}_4-a_3^2\boldsymbol{h}_3 & a_4^2\boldsymbol{h}_4-a_3\boldsymbol{h}_3 \\ a_4\boldsymbol{h}_4-a_3^2\boldsymbol{h}_3 & a_4^2\boldsymbol{h}_4-a_3\boldsymbol{h}_3 \end{bmatrix}$$

Then,

$$\Delta^{1,0}\boldsymbol{b}_{i,j}\times\Delta^{0,1}\boldsymbol{b}_{p,q}=\sum\lambda_{l_1}\left(\boldsymbol{h}_m\times\boldsymbol{h}_n\right)+\sum\lambda_{l_2}\left(\boldsymbol{v}\times\boldsymbol{h}_n\right),\ m=1,2,3;\ n=3,4 \tag{16}$$

In the expression, the parameters λ_{l_1} and λ_{l_2} are the multiply-add combinations of a_i^1 and a_i, which leads to $\lambda_{l_1},\lambda_{l_2}\ge 0$. For the two sets of mixed products in Equation (6), the corresponding values are only determined by a_i^2. It is convenient to control the surface convexity by choosing the values of a_i.

Applied to Construct the Yield Surface Patch

In the field of solid mechanics, the yield surface corresponding to the plastic deformation of materials can be considered to be constructed with Bézier patches. Based on plasticity theory, the yield surface is required to be convex. When constructing a yield surface under plane stress condition in a general stress space, the form of the yield surface is represented by a quasi-ellipsoid. In this case, point A in Figure 4 is the pole along z direction, and B, C and their normals are on the XOY plane, which makes $\boldsymbol{h}_m\times\boldsymbol{h}_n\ /\!/\ \boldsymbol{v}$ and $\left(\boldsymbol{h}_m,\boldsymbol{h}_n,\boldsymbol{h}_k\right)=0$.

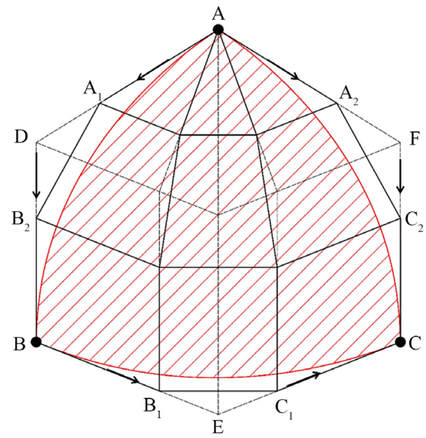

Figure 4. A degenerate Bézier (3 × 3) surface patch with planar-meshescontrol grid.

Consider $\left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{2,0}\boldsymbol{b}_{s,t}\right)$, if $\left(\boldsymbol{h}_m, \boldsymbol{h}_n, \boldsymbol{v}\right) < 0$ is guaranteed, compare the first and second row of the matrix $\Delta^{2,0}\boldsymbol{b}_{s,t}$, and based on the positive and negative consistency, they should have a same structure. We get:

$$\begin{cases} a_i \geq 0 \\ a_j^1 \geq 0, i = 1,2;\ j = 3,4 \\ a_i^2 \geq 0 \end{cases} \tag{17}$$

Then, $0 \leq a_1, a_2 \leq 1/2$.

Similarly, consider $\left(\Delta^{1,0}\boldsymbol{b}_{i,j}, \Delta^{0,1}\boldsymbol{b}_{p,q}, \Delta^{0,2}\boldsymbol{b}_{\alpha,\beta}\right)$, we get:

$$\begin{cases} a_i \geq 0 \\ a_1^1 \geq 0, i = 3,4 \\ a_i^2 \geq 0 \end{cases} \tag{18}$$

Then, $0 \leq a_3, a_4 \leq 1/2$. In summary, $0 \leq a_i \leq 1/2, i = 1,2,3,4$.

In order to illustrate applications of the modeling method in constructing plastic yield surface, the yield data of IF steel (see **Table 1**, data after [21]) is chosen to give modeling.

In **Table 1**, $\theta_1 = -\arctan\left(\dfrac{r_0}{r_0+1}\right)$, $\theta_2 = \dfrac{\pi}{2} + \arctan\left(\dfrac{r_{90}+1}{r_{90}}\right)$, $r_0 = 1.85$, $r_{90} = 2.51$, $\theta_3 = \arctan 0.77$, and r_0, r_{90} represent the anisotropy coefficients along 0˚ and 90˚, respectively.

As shown in **Figure 5**, The data above can construct two interpolated surfaces, one with $f_{UN,0}$, f_{BI} and $f_{SH,0}$ as the interpolation points and the other with $f_{UN,90}$, f_{BI} and $f_{SH,0}$ as the interpolation points. The corresponding tangential directions (perpendicular to the normals) of these interpolation points represent the direction of the edge of the control polygon at the interpolation point. The choice for a_i can be obtained based on existing conditions. For example, the yield points of $f_{UN,45}$ are on the surface which can be used to determine the value of a_1 or a_2; the C1 continuous splicing condition of the two surface patches requires that the sides on both sides of the common point be collinear and of equal length; if there is no limit conditions, take $a_i = 1/3$ which is obtained when the Bézier curve is raised from the second order to the third order. The value scheme of a_i is shown in **Table 2.**

The final results are shown in **Figure 6** and **Figure 7**.

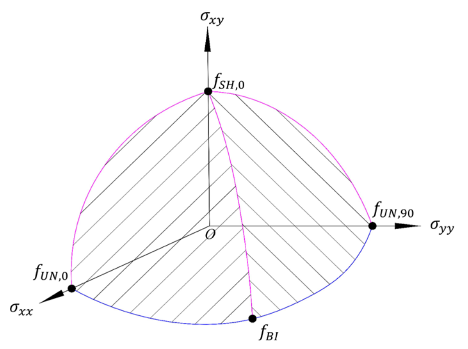

Figure 5. Schematic of interpolated yield surface patches.

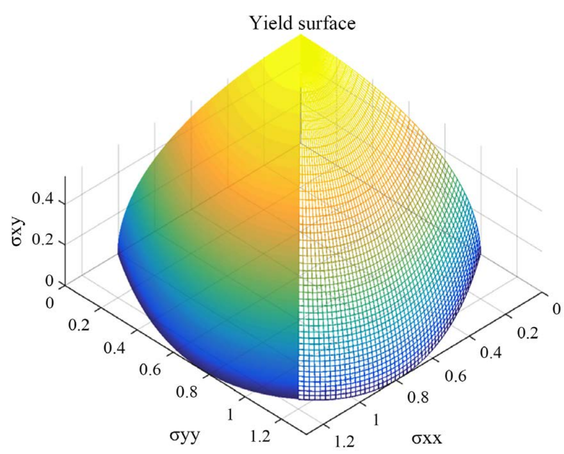

Figure 6. Interpolated yield surface patches.

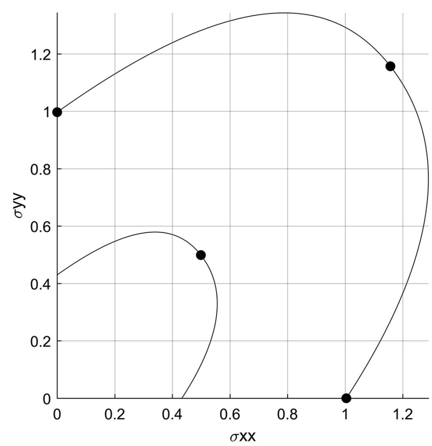

Figure 7. Yield loci of $\sigma_{xy} = 0$ and $\sigma_{xy} = \sigma_{xy,45}$.

Table 1. Normalized yield stress and normal of IF steel (data after [21]).

Yield stress (normalized)	$f_{UN,0} = 1.004$ $(f_{UN,0}, 0, 0)$	$f_{UN,45} = 0.998$ $\left(\frac{f_{UN,45}}{2}, \frac{f_{UN,45}}{2}, \frac{f_{UN,45}}{2}\right)$	$f_{UN,90} = 0.997$ $(0, f_{UN,90}, 0)$	$f_{BI} = 1.157$ $(f_{BI}, f_{BI}, 0)$	$f_{SH,0} = 0.537$ $(0, 0, f_{SH,0})$
Normal (normalized)	$(\cos\theta_1, \sin\theta_1, 0)$	—	$(\cos\theta_2, \sin\theta_2, 0)$	$(\cos\theta_3, \sin\theta_3, 0)$	$(0,0,1)$

Table 2. The value scheme of a_i.

a_i	a_1	a_2	a_3	a_4
Patch 1	0.3691	0.3691	0.3333	0.3046
Patch 2	0.3691	0.3691	0.3333	0.3333

5. Conclusions

A sufficient condition on convexity of parametric Bézier surface patches is found and proved. It associates surface convexity with control grid convexity. This proof makes the conclusion that the convex Bézier grid leads to convex Bézier surface no longer limit to translational surfaces, which provides greater flexibility and convenience for convex surface modeling.

Two cases for interpolation-type surface modeling are introduced as the applications of the new convexity condition. The first case introduces the effect of the position of a free vertex on the convexity of a general Bézier surface patch. The second case takes the degenerate surface as an example and establishes a planar mesh model. The selections of the model parameters determine whether the surface patch is convex or not.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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