British Journal of Mathematics & Computer Science
4(5): 685-703, 2014

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A Strong Method for Symmetric Homogeneous Polynomial Inequalities of Degree Six in Nonnegative Real Variables

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Original Research Article

Received: 23 September 2013
Accepted: 05 November 2013
Published: 05 December 2013

Abstract

Let $f_6(x, y, z)$ be a symmetric homogeneous polynomial of degree six. Based on cancelling the high coefficient of $f_6(x, y, z)$, we give some practical sufficient conditions to have $f_6(x, y, z) \geq 0$ for any nonnegative real variables x, y, z. Some applications are given in order to emphasize the effectiveness of the proposed method.

Keywords: Symmetric homogeneous inequality; Sixth degree polynomial; Sufficient conditions; Highest coefficient; Nonnegative real variables.

2010 Mathematics Subject Classification: 26D05

1 Introduction

A symmetric and homogeneous polynomial of degree six can be written as follows:

$$f_6(x,y,z) = A_1 \sum x^6 + A_2 \sum xy(x^4+y^4) + A_3 \sum x^2y^2(x^2+y^2) + A_4 \sum x^3y^3 + A_5 xyz \sum x^3 + A_6 xyz \sum xy(x+y) + 3A_7x^2y^2z^2, \tag{1.1}$$

where $A_1, ..., A_7$ are real coefficients, and $\sum$ denotes a cyclic sum over x, y, z. In terms of

$$p = x+y+z, \quad q = xy+yz+zx, \quad r = xyz,$$

it can be rewritten in the form

$$f_6(x,y,z) = Ar^2 + g_1(p,q)r + g_2(p,q), \tag{1.2}$$

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where

$$g_1(p,q) = Bp^3 + Cpq, \quad g_2(p,q) = Dp^6 + Ep^4q + Fp^2q^2 + Gq^3,$$

A, B, C, D, E, F, G being real constants. Throughout this paper, as in [1] and [2], we call the constant A the *highest coefficient* of $f_6(x,y,z)$. Since the highest coefficients of the polynomials

$$\sum x^6, \quad \sum xy(x^4+y^4), \quad \sum x^2y^2(x^2+y^2), \quad \sum x^3y^3, \quad xyz\sum x^3, \quad xyz\sum xy(x+y)$$

are, respectively,

$$3, \quad -3, \quad -3, \quad 3, \quad 3, \quad -3,$$

the highest coefficient of $f_6(x,y,z)$ is

$$A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

Any inequality $f_6(x,y,z) \geq 0$ which holds for all nonnegative real numbers x,y,z can be proved by using the following two theorems in [2], which give necessary and sufficient conditions to have $f_6(x,y,z) \geq 0$ for all $x,y,z \geq 0$.

Theorem 1.1. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six which has the highest coefficient $A \leq 0$. The inequality*

$$f_6(x,y,z) \geq 0$$

holds for all nonnegative real numbers x,y,z if and only if $f_6(x,1,1) \geq 0$ and $f_6(0,y,z) \geq 0$ for all $x,y,z \geq 0$.

Theorem 1.2. *Let $f_6(x,y,z)$ be a sixth degree symmetric homogeneous polynomial written in terms of $p = x+y+z$, $q = xy+yz+zx$, $r = xyz$ as follows*

$$f_6(x,y,z) = Ar^2 + g_1(p,q)r + g_2(p,q), \quad A > 0,$$

and let

$$h(t) = 2At + g_1(t+2, 2t+1),$$
$$d(p,q) = g_1^2(p,q) - 4Ag_2(p,q),$$
$$d(p,q) > 0 \iff \frac{p^2}{q} \in \mathbb{I} \cup \mathbb{J},$$

where $\mathbb{I}$ is a union of intervals $\mathbb{I}_i \subseteq [3,4)$, and $\mathbb{J}$ is a union of intervals $\mathbb{J}_i \subseteq [4,\infty]$. The inequality

$$f_6(x,y,z) \geq 0$$

holds for all $x,y,z \geq 0$ if and only if the following three conditions are fulfilled:

(a) *$f_6(x,1,1) \geq 0$ and $f_6(0,y,z) \geq 0$ for all $x,y,z \geq 0$;*

(b) *for each interval $\mathbb{I}_i$, we have $h(t) \geq 0$ for $t \in \mathbb{K}_i$ or $h(t) \leq 0$ for $t \in \mathbb{L}_i$, where*

$$t \in \mathbb{K}_i \iff \frac{(t+2)^2}{2t+1} \in \mathbb{I}_i, \ 0 < t \leq 1,$$

$$t \in \mathbb{L}_i \iff \frac{(t+2)^2}{2t+1} \in \mathbb{I}_i, \ 1 \leq t < 4;$$

(c) *for each interval $\mathbb{J}_i$, we have $g_1(\sqrt{w},1) \geq 0$ for $w \in \mathbb{J}_i$ or $h(t) \leq 0$ for $t \in \mathbb{M}_i$, where*

$$t \in \mathbb{M}_i \iff \frac{(t+2)^2}{2t+1} \in \mathbb{J}_i, \ t \geq 4.$$

In the more difficult case $A > 0$, to prove an inequality $f_6(x,y,z) \geq 0$ using the necessary and sufficient conditions from Theorem 1.2, we need to write the polynomial $f_6(x,y,z)$ in the form (1.2), and this is one of the reasons why the method is rather tedious and laborious (see [2]). In this paper, we present some strong practical sufficient conditions to have $f_6(x,y,z) \geq 0$ for all nonnegative real numbers x,y,z, which can be applied to prove many such inequalities in a much simpler way, without using the functions $g_1(p,q)$ and $g_2(p,q)$ in (1.2), but only the highest coefficient A.

2 Main Results

To obtain the desired results, we need the following lemma in [2].

Lemma 2.1. *Let $x \le y \le z$ be nonnegative real numbers such that $x+y+z=p$ and $xy+yz+zx=q$, where p and q are given nonnegative real numbers satisfying $p^2 \ge 3q \ge 0$. Then, the product $r=xyz$ is maximal when $x=y$, and is minimal when $y=z$ (for $p^2 \le 4q$) or $x=0$ (for $p^2 > 4q$).*

In addition, we will use the following two ideas:

(1) to find a nonnegative symmetric homogeneous function $F_6(x,y,z)$ having the form

$$F_6(x,y,z)=g(p,q)r+h(p,q)$$

and satisfying

$$F_6(x,y,z) \le f_6(x,y,z)$$

for all nonnegative real x,y,z;

(2) to consider successively the cases $p^2 \le 4q$ - when the inequality $F_6(x,y,z) \ge 0$ holds for all nonnegative real numbers x,y,z if and only if $F_6(x,1,1) \ge 0$ for all $x \in [0,4]$, and $p^2 > 4q$ - when the inequality $F_6(x,y,z) \ge 0$ holds if and only if $F_6(x,1,1) \ge 0$ and $F_6(0,y,z) \ge 0$ for all $x > 4$ and $y,z \ge 0$.

Let us define the following nonnegative rational functions

$$f_{\alpha,\beta}(x)=\frac{4(x-1)^4(x-\alpha)^2(x-\beta)^2}{9(4-\alpha-\beta-2\alpha\beta)^2(x+2)^2},$$

$$\bar{f}_{\gamma,\delta}(y,z)=\frac{[2(y+z)^4-(10+\gamma+\delta)yz(y+z)^2+2(2+\gamma)(2+\delta)y^2z^2]^2}{9(4-\gamma-\delta-2\gamma\delta)^2(y+z)^2},$$

where α, β, γ, δ are real numbers such that $\alpha+\beta+2\alpha\beta \ne 4$ and $\gamma+\delta+2\gamma\delta \ne 4$.

For $\beta=-2$, $\beta=-1$, $\beta=0$, $\beta=1$ and $\beta \to \infty$, we get in succession:

$$f_{\alpha,-2}(x)=\frac{4(x-1)^4(x-\alpha)^2}{81(2+\alpha)^2},$$

$$f_{\alpha,-1}(x)=\frac{4(x+1)^2(x-1)^4(x-\alpha)^2}{9(5+\alpha)^2(x+2)^2},$$

$$f_{\alpha,0}(x)=\frac{4x^2(x-1)^4(x-\alpha)^2}{9(4-\alpha)^2(x+2)^2},$$

$$f_{\alpha,1}(x)=\frac{4(x-1)^6(x-\alpha)^2}{81(1-\alpha)^2(x+2)^2},$$

$$f_{\alpha,\infty}(x)=\frac{4(x-1)^4(x-\alpha)^2}{9(1+2\alpha)^2(x+2)^2}.$$

Also, for $\delta=-2$, $\delta=-1$, $\delta=0$, $\delta=1$ and $\delta \to \infty$, we get

$$\bar{f}_{\gamma,-2}(y,z)=\frac{(y+z)^2[2(y+z)^2-(8+\gamma)yz]^2}{81(2+\gamma)^2},$$

$$\bar{f}_{\gamma,-1}(y,z)=\frac{[2(y+z)^4-(9+\gamma)yz(y+z)^2+2(2+\gamma)y^2z^2]^2}{9(5+\gamma)^2(y+z)^2},$$

$$\bar{f}_{\gamma,0}(y,z)=\frac{(y-z)^4[2(y+z)^2-(2+\gamma)yz]^2}{9(4-\gamma)^2(y+z)^2},$$

$$\bar{f}_{\gamma,1}(y,z)=\frac{[2(y+z)^4-(11+\gamma)yz(y+z)^2+6(2+\gamma)y^2z^2]^2}{81(1-\gamma)^2(y+z)^2},$$

$$\bar{f}_{\gamma,\infty}(y,z) = \frac{y^2z^2[(y+z)^2 - 2(2+\gamma)yz]^2}{9(1+2\gamma)^2(y+z)^2}.$$

In particular,

$$f_{-2,\infty}(x) = \frac{4(x-1)^4}{81}, \quad \bar{f}_{-2,\infty}(y,z) = \frac{y^2z^2(y+z)^2}{81},$$

$$f_{0,\infty}(x) = \frac{4x^2(x-1)^4}{9(x+2)^2}, \quad \bar{f}_{0,\infty}(y,z) = \frac{y^2z^2(y-z)^4}{9(y+z)^2},$$

$$f_{1,\infty}(x) = \frac{4(x-1)^6}{81(x+2)^2}, \quad \bar{f}_{1,\infty}(y,z) = \frac{y^2z^2(y^2+z^2-4yz)^2}{81(y+z)^2},$$

$$f_{\infty,\infty}(x) = \frac{(x-1)^4}{9(x+2)^2}, \quad \bar{f}_{\infty,\infty}(y,z) = \frac{y^4z^4}{9(y+z)^2}.$$

Let us still define the nonnegative functions

$$f_\delta(x) = \left[x - \frac{\delta(2x+1)^2}{x+2}\right]^2, \quad \bar{f}_\delta(y,z) = \frac{\delta^2y^4z^4}{(y+z)^2},$$

where δ is a real number. For $\delta = 1/3$ and $\delta = 0$, we get

$$f_{1/3}(x) = \frac{(x-1)^4}{9(x+2)^2}, \quad \bar{f}_{1/3}(y,z) = \frac{y^4z^4}{9(y+z)^2},$$

$$f_0(x) = x^2, \quad \bar{f}_0(y,z) = 0.$$

The following two theorems are useful to prove symmetric homogeneous polynomial inequalities of sixth degree in nonnegative real variables x, y, z and having the highest coefficient nonnegative, especially in the cases where the equality occurs for $(1,1,1)$, and/or $(\alpha,1,1)$, and/or $(\beta,1,1)$, and/or $(1,0,0)$, where α and β are nonnegative real numbers.

Theorem 2.2. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A \ge 0$. The inequality $f_6(x,y,z) \ge 0$ holds for all $x,y,z \ge 0$ if there exist four real numbers α, β, γ, δ such that the following two conditions are fulfilled:*

(*a*) $f_6(x,1,1) \ge Af_{\alpha,\beta}(x)$ *for* $0 \le x \le 4$;

(*b*) $f_6(x,1,1) \ge Af_{\gamma,\delta}(x)$ *and* $f_6(0,y,z) \ge A\bar{f}_{\gamma,\delta}(y,z)$ *for* $x > 4,\ y \ge 0,\ z \ge 0.$

Theorem 2.3. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A \ge 0$. The inequality $f_6(x,y,z) \ge 0$ holds for all $x,y,z \ge 0$ if the following two conditions are fulfilled:*

(*a*) *there exist two real numbers α and β such that*

$$f_6(x,1,1) \ge Af_{\alpha,\beta}(x) \ \ for \ \ 0 \le x \le 4;$$

(*b*) *there exists a real numbers δ such that*

$$f_6(x,1,1) \ge Af_\delta(x) \ \ and \ \ f_6(0,y,z) \ge A\bar{f}_\delta(y,z) \ \ for \ \ x > 4,\ y \ge 0,\ z \ge 0.$$

Notice that the relative degree of the rational functions $f_{\gamma,\delta}(x)$, $f_{\gamma,\infty}(x)$ and $f_{\infty,\infty}(x)$ are six, four and two, respectively. Also, $f_\delta(x)$ has the relative degree two.

Next, we will apply Theorem 2.2 and Theorem 2.3 to prove six strong and sharp symmetric homogeneous polynomial inequalities of degree six in nonnegative real variables, which were posted on the known website Art of Problem Solving (see [3]...[6]).

Proposition 2.1. *Let x, y, z be nonnegative real numbers. If $k \le 4$, then*

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \ge (5-3k)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$ - see [3].

Proposition 2.2. *Let x, y, z be nonnegative real numbers. If k is a real numbers, then*

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) \ge 0,$$

with equality for $x = y = z$, for $y = z = 0$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$ - see [4].

Proposition 2.3. *Let x, y, z be nonnegative real numbers. For any real k, we have*

$$\sum x(y+z)(x-y)(x-z)(x-ky)(x-kz) + (k-3)(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), for $y = z = 0$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation)) if $k > 0$ - see [4].

Proposition 2.4. *Let x, y, z be nonnegative real numbers, and let*

$$\alpha_k = \begin{cases} 4(k-2), & k \le 6 \\ \dfrac{(k+2)^2}{4}, & k \ge 6 \end{cases}.$$

Then,

$$\sum x(x-y)(x-z)(x-ky)(x-kz) + \frac{\alpha_k (x-y)^2(y-z)^2(z-x)^2}{x+y+z} \ge 0,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$, and for $x = 0$ and $y/z + z/y = (k-2)/2$ (or any cyclic permutation) if $k > 6$ - see [5].

Proposition 2.5. *Let x, y, z be nonnegative real numbers, and let*

$$\alpha_k = \begin{cases} 3(1-k), & k \le 0 \\ 3+k, & k \ge 0 \end{cases}.$$

If $k \in (-\infty, -5/4] \cup \{0, 1, 2, 3\}$, then

$$\sum (x-y)(x-z)(x-ky)(x-kz) \ge \frac{\alpha_k (x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx},$$

with equality for $x = y = z$, for $x = 0$ and $y/z + z/y = 2 + |k|$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$ - see [6].

Proposition 2.6. *Let x, y, z be nonnegative real numbers. If k is a real numbers such that $|k| \ge 1$, then*

$$\sum (y+z)(x-y)(x-z)(x-ky)(x-kz) \ge \frac{(2+|k|)^2(x-y)^2(y-z)^2(z-x)^2}{x+y+z},$$

with equality for $x = y = z$, for $x/k = y = z$ (or any cyclic permutation) if $k > 0$, for $y = z = 0$ (or any cyclic permutation), and for $x = 0$ and $y/z + z/y = 2 + 2|k|$ (or any cyclic permutation) - see [5].

Remark 2.1. The coefficients of the product $(x-y)^2(y-z)^2(z-x)^2$ in Propositions 2.1 ... 2.6 are the best possible. We can show this as follows.

With regard to Proposition 2.1, setting $x=0$ in the inequality

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \geq \alpha_k(x-y)^2(y-z)^2(z-x)^2$$

yields

$$(y-z)^2[y^4+z^4-(k-1)yz(y^2+z^2)+(1-k-\alpha_k)y^2z^2] \geq 0.$$

In addition, setting $y=z=1$ in the necessary condition

$$y^4+z^4-(k-1)yz(y^2+z^2)+(1-k-\alpha_k)y^2z^2 \geq 0$$

provides $\alpha_k \leq 5-3k$.

With regard to Proposition 2.2, putting $x \to \infty$ in the inequality

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) \geq \alpha_k(x-y)^2(y-z)^2(z-x)^2$$

leads to

$$yz \geq \alpha_k(y-z)^2.$$

This inequality holds for all nonnegative y, z if and only if $\alpha_k \leq 0$.

With regard to Proposition 2.3, making $x=0$ in the inequality

$$\sum x(y+z)(x-y)(x-z)(x-ky)(x-kz) + \alpha_k(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

we get

$$yz(y-z)^2[y^2+z^2-(k-1-\alpha_k)yz] \geq 0.$$

For $y=z=1$, the necessary condition

$$y^2+z^2-(k-1-\alpha_k)yz \geq 0$$

yields $\alpha_k \geq k-3$.

With regard to Proposition 2.4, setting $x=0$ provides

$$(y-z)^2[(y^2+z^2)^2-(k-2)yz(y^2+z^2)+(\alpha_k-2k)y^2z^2] \geq 0.$$

For $y=z=1$, the necessary condition

$$(y^2+z^2)^2-(k-2)yz(y^2+z^2)+(\alpha_k-2k)y^2z^2 \geq 0$$

involves $\alpha_k \geq 4(k-2)$. Also, for

$$y^2+z^2=\frac{k-2}{2}yz, \quad k \geq 6,$$

we get

$$[4\alpha_k-(k+2)^2]y^2z^2 \geq 0,$$

which involves $\alpha_k \geq (k+2)^2/4$.

With regard to Proposition 2.5, setting $x=0$ yields

$$(y^2+z^2)^2-(1+k+\alpha_k)yz(y^2+z^2)+(k^2+2k-2+2\alpha_k)y^2z^2 \geq 0.$$

For $k>0$, choosing y and z such that $y^2+z^2=(2+k)yz$, we get

$$k(\alpha_k-k-3)y^2z^2 \leq 0,$$

which involves $\alpha_k \leq k+3$. Similarly, for $k<0$, choosing $y^2+z^2=(2-k)yz$, we get

$$(-k)(\alpha_k+3k-3)y^2z^2 \leq 0,$$

which provides $\alpha_k \leq 3(1-k)$. Also, for $k=0$, we get

$$(y-z)^2[(y-z)^2+(3-\alpha_0)yz] \geq 0,$$

which yields $\alpha_0 \leq 3$.

With regard to Proposition 2.6, setting $x=0$ in the inequality

$$\sum(y+z)(x-y)(x-z)(x-ky)(x-kz) \geq \frac{\alpha_k(x-y)^2(y-z)^2(z-x)^2}{x+y+z}$$

implies

$$yz[(y^2+z^2)^2-(\alpha_k-k^2)yz(y^2+z^2)+2(\alpha_k+k^2-2)y^2z^2] \geq 0.$$

For $y^2+z^2=2(1+|k|)yz$, from the necessary condition

$$(y^2+z^2)^2-(\alpha_k-k^2)yz(y^2+z^2)+2(\alpha_k+k^2-2)y^2z^2 \geq 0,$$

we get

$$|k|[\alpha_k-(2+|k|)^2]y^2z^2 \leq 0,$$

which yields $\alpha_k \leq (2+|k|)^2$.

Remark 2.2. For $k=4$, the inequality in Proposition 2.1 turns into

$$\sum x^2(x-y)(x-z)(x-4y)(x-4z)+7(x-y)^2(y-z)^2(z-x)^2 \geq 0,$$

which is equivalent to

$$(x^2+y^2+z^2-xy-yz-zx)(x^2+y^2+z^2-2xy-2yz-2zx)^2 \geq 0.$$

The equality occurs when $x=y=z$, and when $\sqrt{x}=\sqrt{y}+\sqrt{z}$ (or any cyclic permutation).

For $k=0$, the inequality in Proposition 2.2 turns into

$$xyz\sum x(x-y)(x-z) \geq 0.$$

The equality holds for $x=y=z$, and also for $x=0$ (or any cyclic permutation).

Remark 2.3. The inequality in Proposition 2.1 can be extended for $k \geq 4$, as follows

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \geq \frac{20+12k-4k^2-k^4}{4(k-1)^2}(x-y)^2(y-z)^2(z-x)^2.$$

Also, the inequalities in Proposition 2.5 and Proposition 2.6 hold for all $k \in \mathbb{R}$. These are very difficult inequalities which cannot be proved using Theorem 2.2 and Theorem 2.3. We will prove them in a future paper using a similar but more powerful method focused on this type of inequalities.

Remark 2.4. Substituting $k-1$ for k in Propositions 2.1 and 2.2, and using then the identities

$$\sum x^2(x-y)(x-z)[x-(k-1)y](x-(k-1)z] =$$
$$= \sum x^2(x-y)(x-z)(x-ky+z)(x-kz+y)+k(x-y)^2(y-z)^2(z-x)^2,$$

$$\sum yz(x-y)(x-z)[x-(k-1)y](x-(k-1)z] =$$
$$= \sum yz(x-y)(x-z)(x-ky+z)(x-kz+y)+k(x-y)^2(y-z)^2(z-x)^2,$$

we get the following equivalent propositions.

Proposition 2.7. *Let x, y, z be nonnegative real numbers. If $k \le 5$, then*

$$\sum x^2(x-y)(x-z)(x-ky+z)(x-kz+y) \ge 4(2-k)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/(k-1) = y = z$ (or any cyclic permutation) if $1 < k \le 5$.

Proposition 2.8. *Let x, y, z be nonnegative real numbers. If k is a real number, then*

$$\sum yz(x-y)(x-z)(x-ky+z)(x-kz+y) + k(x-y)^2(y-z)^2(z-x)^2 \ge 0,$$

with equality for $x = y = z$, for $y = z = 0$ (or any cyclic permutation), and for $x/(k-1) = y = z$ (or any cyclic permutation) if $k > 1$.

3 Proof of Theorem 2.2

Recall the notation

$$p = x + y + z\ ,\quad q = xy + yz + zx\ ,\quad r = xyz.$$

The main idea is to find two *nonnegative* symmetric homogeneous functions $h_1(x, y, z)$ and $h_2(x, y, z)$ of sixth degree and having the form $r^2 + g(p, q)r + h(p, q)$, such that

(A1) $f_6(x, y, z) \ge Ah_1(x, y, z)$ for all nonnegative real x, y, z satisfying $p^2 \le 4q$;

(A2) $f_6(x, y, z) \ge Ah_2(x, y, z)$ for all nonnegative real x, y, z satisfying $p^2 > 4q$.

Clearly, if the conditions (A1) and (A2) are fulfilled, then $f_6(x, y, z) \ge 0$ for all nonnegative real x, y, z. Let us denote

$$H_1(x, y, z) = f_6(x, y, z) - Ah_1(x, y, z), \quad H_2(x, y, z) = f_6(x, y, z) - Ah_2(x, y, z).$$

Since the functions $H_1(x, y, z)$ and $H_2(x, y, z)$ have the highest coefficient equal to zero, we can apply Lemma 2.1 to analyse the conditions (A1) and (A2), which are respectively equivalent to

(B1) $H_1(x, y, z) \ge 0$ for all nonnegative real x, y, z satisfying $p^2 \le 4q$;

(B2) $H_2(x, y, z) \ge 0$ for all nonnegative real x, y, z satisfying $p^2 > 4q$.

For fixed p and q, the inequalities from (B1) and (B2) can be written as $g_1(r) \ge 0$ and $g_2(r) \ge 0$, where g_1 and g_2 are linear functions, which are minimal when r is minimal or maximal. Thus, by Lemma 2.1 and due to symmetry, the inequality $H_1(x, y, z) \ge 0$ holds for $p^2 \le 4q$ if it holds for $y = z$, while the inequality $H_2(x, y, z) \ge 0$ holds for $p^2 > 4q$ if it holds for $y = z$ and for $x = 0$.

On the other hand, due to homogeneity, we may reduce the case $y = z \ne 0$ to $y = z = 1$, when the conditions $p^2 \le 4q$ and $p^2 > 4q$ are equivalent to $x \in [0, 4]$ and $x > 4$, respectively. Thus, the conditions (B1)and (B2) are fulfilled if and only if the following conditions are satisfied:

(C1) $H_1(x, 1, 1) \ge 0$ for $x \in [0, 4]$;

(C2) $H_2(x, 1, 1) \ge 0$ and $H_2(0, y, z) \ge 0$ for $x > 4$ and $y, z \ge 0$.

We will show that the conditions (C1) and (C2) are satisfied if we choose the nonnegative functions

$$h_1(x, y, z) = \left(r + \frac{2}{a_1}p^3 - \frac{b_1}{a_1}pq + \frac{c_1}{a_1}\cdot\frac{q^2}{p}\right)^2,$$

$$h_2(x, y, z) = \left(r + \frac{2}{a_2}p^3 - \frac{b_2}{a_2}pq + \frac{c_2}{a_2}\cdot\frac{q^2}{p}\right)^2,$$

which satisfy

$$h_1(1, 1, 1) = h_1(\alpha, 1, 1) = h_1(\beta, 1, 1) = 0,$$

$$h_2(1, 1, 1) = h_2(\gamma, 1, 1) = h_2(\delta, 1, 1) = 0.$$

After some calculation, we get

$$a_1 = 3(4-\alpha-\beta-2\alpha\beta), \quad b_1 = 10+\alpha+\beta, \quad c_1 = 2(2+\alpha)(2+\beta),$$

$$a_2 = 3(4-\gamma-\delta-2\gamma\delta), \quad b_2 = 10+\gamma+\delta, \quad c_2 = 2(2+\gamma)(2+\delta).$$

We can check that

$$h_1(x,1,1) = f_{\alpha,\beta}(x), \quad h_2(x,1,1) = f_{\gamma,\delta}(x), \quad h_2(0,y,z) = \bar{f}_{\gamma,\delta}(x).$$

According to the hypotheses in Theorem 2.2, we have

$$H_1(x,1,1) = f_6(x,1,1) - Ah_1(x,1,1) = f_6(x,1,1) - Af_{\alpha,\beta}(x) \geq 0 \text{ for } x \in [0,4],$$

$$H_2(x,1,1) = f_6(x,1,1) - Ah_2(x,1,1) = f_6(x,1,1) - Af_{\gamma,\delta}(x) \geq 0 \text{ for } x > 4,$$

$$H_2(0,y,z) = f_6(0,y,z) - Ah_2(0,y,z) = f_6(0,y,z) - A\bar{f}_{\gamma,\delta}(y,z) \geq 0 \text{ for } y,z \geq 0.$$

This completes the proof.

Remark 3.1. In addition to the relations

$$h_2(1,1,1) = h_2(\gamma,1,1) = h_2(\delta,1,1) = 0,$$

we have also

$$h_2(0,u_2,v_2) = 0$$

for all nonnegative u_2 and v_2 satisfying

$$\frac{u_2}{v_2} + \frac{v_2}{u_2} = \frac{\gamma+\delta+2 \pm \sqrt{(\gamma+\delta+10)^2 - 16(\gamma+2)(\delta+2)}}{4}.$$

Remark 3.2. For $\delta = -2$, we get

$$h_2(x,y,z) = \left(r + \frac{2}{9\gamma+18}p^3 - \frac{\gamma+8}{9\gamma+18}pq\right)^2.$$

This function is zero for $(x,y,z) = (1,1,1)$, $(x,y,z) = (\gamma,1,1)$ and $(x,y,z) = (0,u,v)$, where

$$\frac{u}{v} + \frac{v}{u} = 2 + \frac{\gamma}{2}.$$

In addition, if $\gamma \to \infty$, then

$$h_2(x,y,z) = \left(r - \frac{1}{9}pq\right)^2.$$

Obviously, $h_2(x,y,z)$ is zero for $(x,y,z) = (1,1,1)$ and $(x,y,z) = (1,0,0)$.

For $\delta \to \infty$, we get

$$h_2(x,y,z) = \left(r + \frac{1}{6\gamma+3}pq - \frac{2\gamma+4}{6\gamma+3} \cdot \frac{q^2}{p}\right)^2.$$

In this case, $h_2(x,y,z)$ is zero for $(x,y,z) = (1,1,1)$, $(x,y,z) = (\gamma,1,1)$, $(x,y,z) = (1,0,0)$ and $(x,y,z) = (0,u,v)$, where

$$\frac{u}{v} + \frac{v}{u} = 2 + 2\gamma.$$

In addition, if $\gamma \to \infty$, then we have

$$h_2(x,y,z) = \left(r - \frac{1}{3} \cdot \frac{q^2}{p}\right)^2.$$

Clearly, $h_2(x,y,z)$ is zero for $(x,y,z) = (1,1,1)$ and $(x,y,z) = (1,0,0)$.

4 Proof of Theorem 2.3

The proof is similar to the one of Theorem 2.2. However, here we choose

$$h_2(x,y,z) = \left(r - \frac{\delta q^2}{p}\right)^2,$$

where $p = x+y+z$, $q = xy+yz+zx$, $r = xyz$. Since

$$h_2(x,1,1) = f_\delta(x), \quad h_2(0,y,z) = \bar{f}_\delta(y,z),$$

from the hypothesis (b), we have

$$H_2(x,1,1) = f_6(x,1,1) - Ah_2(x,1,1) = f_6(x,1,1) - Af_\delta(x) \geq 0 \text{ for } x > 4$$

and

$$H_2(0,y,z) = f_6(0,y,z) - Ah_2(0,y,z) = f_6(0,y,z) - A\bar{f}_\delta(y,z) \geq 0 \text{ for } y,z \geq 0.$$

5 Proof of Propositions 2.1 ... 2.6

Proof of Proposition 2.1. Denote

$$f(x,y,z) = \sum x^2(x-y)(x-z)(x-ky)(x-kz),$$

and write the desired inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = f(x,y,z) - (5-3k)(x-y)^2(y-z)^2(z-x)^2.$$

From

$$x(x-y)(x-z) = 2x^3 - px^2 + r$$

and

$$x(x-ky)(x-kz) = x^3 - kx^2(y+z) + k^2r = (k+1)x^3 - kpx^2 + k^2r,$$

it follows that $f(x,y,z)$ has the same highest coefficient A_1 as $g(x,y,z)$, where

$$\begin{aligned} g(x,y,z) &= \sum(2x^3+r)[(k+1)x^3+k^2r] \\ &= 2(k+1)\sum x^6 + (2k^2+k+1)r\sum x^3 + 3k^2r^2; \end{aligned}$$

that is,

$$A_1 = 6(k+1) + 3(2k^2+k+1) + 3k^2 = 9(k^2+k+1).$$

Since the highest coefficient of the product $(x-y)^2(y-z)^2(z-x)^2$ is equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A = A_1 + 27(5-3k) = 9(4-k)^2.$$

On the other hand, we have

$$f_6(x,1,1) = x^2(x-1)^2(x-k)^2,$$

$$\begin{aligned} f_6(0,y,z) &= (y-z)^4[y^2+z^2+(3-k)yz] + (5-3k-\alpha_k)y^2z^2(y-z)^2 \\ &= (y-z)^4[(y-z)^2+(5-k)yz]. \end{aligned}$$

For $k=4$, we have $A=0$. According to Theorem 1.1 (or Theorem 2.2), the desired inequality is true since $f_6(x,1,1) \geq 0$ and $f_6(0,y,z) \geq 0$ for all $x,y,z \geq 0$.

For $k < 4$, we have $A > 0$. To prove the desired inequality, we will apply Theorem 2.3 for

$$\alpha = k, \quad \beta = 0, \quad \delta = 0,$$

when

$$f_{k,0}(x) = \frac{4x^2(x-1)^4(x-k)^2}{9(4-k)^2(x+2)^2}, \quad f_0(x) = x^2, \quad \bar{f}_0(y,z) = 0.$$

The condition (a) is fulfilled since

$$f_6(x,1,1) - Af_{k,0} = \frac{3x^3(x-1)^2(x-k)^2(4-x)}{(x+2)^2} \geq 0$$

for $0 \leq x \leq 4$.

The condition (b) is satisfied if $f_6(x,1,1) \geq Ax^2$ and $f_6(0,y,z) \geq 0$ for $x > 4$, $y \geq 0$, $z \geq 0$. Indeed,

$$f_6(x,1,1) - Ax^2 = x^2[(x-1)^2(x-k)^2 - 9(4-k)^2] > 0,$$

since $(x-1)^2 > 9$ and $(x-k)^2 > (4-k)^2$ for $x > 4$ and $k < 4$. Also, the condition $f_6(0,y,z) \geq 0$ is clearly true.

□

Proof of Proposition 2.2. If one of x, y, z is zero, the inequality is trivial. On the other hand, the inequality remains unchanged by replacing x, y, z and k with $1/x, 1/y, 1/z$ and $1/k$, respectively. Therefore, it suffices to consider only the cases $0 \leq k \leq 1$ and $k \leq -1$.

Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = \sum yz(x-y)(x-z)(x-ky)(x-kz).$$

Since

$$(x-y)(x-z) = x^2 + 2yz - q$$

and

$$(x-ky)(x-kz) = x^2 + (k^2+k)yz - kq,$$

$f_6(x,y,z)$ has the same highest coefficient as

$$g(x,y,z) = \sum yz(x^2+2yz)[x^2+(k^2+k)yz]$$
$$= 2(k^2+k)\sum y^3z^3 + xyz\sum x^3 + 3(k^2+k+2)x^2y^2z^2;$$

that is,

$$A = 6(k^2+k) + 3 + 3(k^2+k+2) = 9(k^2+k+1).$$

On the other hand,

$$f_6(x,1,1) = (x-1)^2(x-k)^2, \quad f_6(0,y,z) = k^2y^3z^3.$$

Next, we apply Theorem 2.3 for $\alpha = k$ and $\beta = \infty$, when

$$f_{k,\infty}(x) = \frac{4(x-1)^4(x-k)^2}{9(2k+1)^2(x+2)^2}.$$

Since

$$f_6(x,1,1) - Af_{k,\infty} = \frac{(x-1)^2(x-k)^2[(2k+1)^2(x+2)^2 - 4(k^2+k+1)(x-1)^2]}{(2k+1)^2(x+2)^2},$$

the condition (a) in Theorem 2.3 is fulfilled if

$$(2k+1)^2(x+2)^2 \geq 4(k^2+k+1)(x-1)^2$$

for $0 \le x \le 4$. This is true because $(2k+1)^2 \ge k^2+k+1$ and $(x+2)^2 \ge 4(x-1)^2$.

In order to prove the condition (b), we consider two cases: $0 \le k \le 1$ and $k \le -1$.

Case 1: $0 \le k \le 1$. We choose

$$\delta = \frac{2k}{3\sqrt{k^2+k+1}},$$

when

$$f_6(0,y,z) - A\bar{f}_\delta(y,z) = y^3z^3\left[k^2 - \frac{9(k^2+k+1)\delta^2 yz}{(y+z)^2}\right]$$
$$\ge y^3z^3\left[k^2 - \frac{9(k^2+k+1)\delta^2}{4}\right] = 0.$$

The condition $f_6(x,1,1) \ge Af_\delta(x)$ is equivalent to

$$(x-1)^2(x-k)^2(x+2)^2 \ge [3K(x^2+2x) - 2k(2x+1)^2]^2,$$

where $K = \sqrt{k^2+k+1}$. Since $0 \le k \le 1$ and $x > 4$, we need to show that

$$(x-1)(x-k)(x+2) \ge 2k(2x+1)^2 - 3K(x^2+2x) \tag{5.1}$$

and

$$(x-1)(x-k)(x+2) \ge 3K(x^2+2x) - 2k(2x+1)^2. \tag{5.2}$$

Write the inequality (5.1) as $xf(x) \ge 0$, where

$$f(x) = x^2 + (3K - 9k + 1)x + 6K - 9k - 2.$$

Since

$$f(x) = (x-4)^2 + 3[K + 3(1-k)](x-4) + 9(2K - 5k + 2) > 9(2K - 5k + 2),$$

it suffices to prove that $2K - 5k + 2 \ge 0$. Indeed, we have

$$2K - 5k + 2 > \frac{3(k+1)}{2} - 5k + 2 = \frac{7(1-k)}{2} \ge 0.$$

Since $k+1 \ge K$, the inequality (5.2) is true if

$$(x-1)(x-k)(x+2) \ge 3(k+1)(x^2+2x) - 2k(2x+1)^2,$$

which is equivalent to

$$x^3 + 2(2k-1)x^2 + (k-8)x + 4k \ge 0,$$
$$x^2(x-4) + 2(2k+1)x(x-4) + 17kx + 4k \ge 0.$$

Clearly, the last inequality is true.

Case 2: $k \le -1$. We choose

$$\delta = \frac{-k}{3\sqrt{k^2+k+1}},$$

when

$$f_6(0,y,z) - A\bar{f}_\delta(y,z) = y^3z^3\left[k^2 - \frac{9(k^2+k+1)\delta^2 yz}{(y+z)^2}\right]$$
$$\ge y^3z^3\left[k^2 - \frac{9(k^2+k+1)\delta^2}{4}\right] = \frac{3}{4}k^2y^3z^3 \ge 0.$$

The condition $f_6(x,1,1) \ge Af_\delta(x)$ is equivalent to

$$(x-1)^2(x-k)^2(x+2)^2 \ge [-3K(x^2+2x) - k(2x+1)^2]^2,$$

where $K = \sqrt{k^2+k+1}$. Since $K \le -k$, we have

$$-3K(x^2+2x) - k(2x+1)^2 \ge 3k(x^2+2x) - k(2x+1)^2 = -k(x-1)^2 > 0.$$

Therefore, it suffices to show that

$$(x-1)(x-k)(x+2) \ge -3K(x^2+2x) - k(2x+1)^2.$$

Since $x-k > 4-k > 0$ and $K > -k-1$, it is enough to prove that

$$(x-1)(4-k)(x+2) \ge 3(k+1)(x^2+2x) - k(2x+1)^2,$$

which is equivalent to

$$x^2 - (3k+2)x + 3k - 8 \ge 0.$$

Indeed, for $k \le -1$ and $x > 4$, we get

$$x^2 - (3k+2)x + 3k - 8 > 16 - 4(3k+2) + 3k - 8 = -9k > 0.$$

□

Proof of Proposition 2.3. Denote the left side of the inequality by $f_6(x,y,z)$. From

$$f_6(x,y,z) = \sum x(p-x)(x^2+2yz-q)[x^2+(k^2+k)yz-kq] + (k-3)\prod(x-y)^2,$$

it follows that f_6 has the same highest coefficient A as

$$\begin{aligned} g(x,y,z) &= -\sum x^2(x^2+2yz)[x^2+(k^2+k)yz] - 27(k-3)x^2y^2z^2 \\ &= -6k(k+1)x^2y^2z^2 - (k^2+k+2)xyz\sum x^3 - \sum x^6 - 27(k-3)x^2y^2z^2; \end{aligned}$$

therefore,

$$A = -6k(k+1) - 3(k^2+k+2) - 3 - 27(k-3) = 9(8-4k-k^2).$$

For $k \in (-\infty, -2\sqrt{3}-2] \cup [2\sqrt{3}-2, \infty)$, we have $A \le 0$. According to Theorem 1.1, the desired inequality is true if $f_6(x,1,1) \ge 0$ and $f_6(0,y,z) \ge 0$ for $x,y,z \ge 0$. Indeed,

$$f_6(x,1,1) = 2x(x-1)^2(x-k)^2 \ge 0,$$

$$f_6(0,y,z) = yz(y-z)^4 \ge 0.$$

Consider now that

$$k \in [-2\sqrt{3}-2, 2\sqrt{3}-2],$$

and apply Theorem 2.3 for

$$\alpha = k, \quad \beta = 0, \quad \delta = 0.$$

The condition (a), namely $f_6(x,1,1) \ge Af_{k,0}(x)$ for $0 \le x \le 4$, is satisfied if

$$(4-k)^2(x+2)^2 \ge 2(8-4k-k^2)x(x-1)^2.$$

This is true since $(4-k)^2 \ge 2(8-4k-k^2)$ and $(x+2)^2 \ge x(x-1)^2$. Indeed,

$$(4-k)^2 - 2(8-4k-k^2) = 3k^2 \ge 0,$$

$$(x+2)^2 - x(x-1)^2 = (4-x)(1+x+x^2) \ge 0.$$

The condition (b) is fulfilled if $f_6(x,1,1) \ge Ax^2$ and $f_6(0,y,z) \ge 0$ for $x > 4,\ y \ge 0,\ z \ge 0$. The second condition is clearly true, while the first condition is fulfilled if

$$2(x-1)^2(x-k)^2 \ge 9(8-4k-k^2)x.$$

Since $4(x-1)^2-9x=(x-4)(4x-1)>0$ and $x-k>4-k>0$, we have

$$4(x-1)^2(x-k)^2-18(8-4k-k^2)x>9x(4-k)^2-18(8-4k-k^2)x=27k^2x\geq 0.$$

□

Proof of Proposition 2.4. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=(x+y+z)\sum x(x-y)(x-z)(x-ky)(x-kz)+\alpha_k(x-y)^2(y-z)^2(z-x)^2.$$

Since the product $(x-y)^2(y-z)^2(z-x)^2$ has the highest coefficient equal to -27, $f_6(x,y,z)$ has the highest coefficient

$$A=-27\alpha_k.$$

Also, we have

$$f_6(x,1,1)=x(x+2)(x-1)^2(x-k)^2,$$
$$f_6(0,y,z)=(y-z)^2[(y+z)^4-(k+2)yz(y+z)^2+\alpha_k y^2z^2].$$

There are three cases to consider.

Case 1: $k\geq 6$. Since

$$A=-27\alpha_k=\frac{-27(k+2)^2}{4}<0,$$

the desired inequality is true if $f_6(x,1,1)\geq 0$ and $f_6(0,y,z)\geq 0$ for $x,y,z\geq 0$ (Theorem 1.1). The first condition is clearly true and

$$\begin{aligned}f_6(0,y,z)&=(y-z)^2[(y+z)^4-(k+2)yz(y+z)^2+\frac{(k+2)^2}{4}y^2z^2]\\&=(y-z)^2\left[(y+z)^2-\frac{k+2}{2}yz\right]^2\geq 0.\end{aligned}$$

Case 2: $2\leq k\leq 6$. Since

$$A=-27\alpha_k=-108(k-2)\leq 0,$$

the desired inequality is true if $f_6(x,1,1)\geq 0$ and $f_6(0,y,z)\geq 0$ for $x,y,z\geq 0$. The first condition is true and

$$\begin{aligned}f_6(0,y,z)&=(y-z)^2[(y+z)^4-(k+2)yz(y+z)^2+4(k-2)y^2z^2]\\&=(y-z)^4[(y+z)^2-(k-2)yz]\geq (y-z)^6\geq 0.\end{aligned}$$

Case 3: $k\leq 2$. We have

$$A=-27\alpha_k=108(2-k)\geq 0.$$

We will apply Theorem 2.3 for

$$\alpha=k,\quad \beta=0,\quad \delta=0.$$

Since

$$f_6(x,1,1)-Af_{k,0}(x)=\frac{x(x-1)^2(x-k)^2[(4-k)^2(x+2)^3-48(2-k)x(x-1)^2]}{(4-k)^2(x+2)^2},$$

the condition (a) is true if

$$(4-k)^2(x+2)^3\geq 48(2-k)x(x-1)^2$$

for $0\leq x\leq 4$. This inequality follows by multiplying the inequalities

$$(4-k)^2\geq 8(2-k)$$

and

$$(x+2)^3\geq 6x(x-1)^2,$$

which are equivalent to $k^2 \geq 0$ and $(4-x)(2+2x+5x^2) \geq 0$, respectively.

The condition (b) is fulfilled if $f_6(x,1,1) \geq Ax^2$ and $f_6(0,y,z) \geq 0$ for $x > 4,\ y \geq 0,\ z \geq 0.$ The first condition is equivalent to

$$(x+2)(x-1)^2(x-k)^2 \geq 108(2-k)x.$$

This inequality follows from

$$4(x-1)^2 \geq 9x$$

and

$$(x+2)(x-k)^2 \geq 48(2-k).$$

Indeed,

$$4(x-1)^2 - 9x = (x-4)(4x-1) \geq 0,$$
$$(x+2)(x-k)^2 - 48(2-k) \geq 6(4-k)^2 - 48(2-k) = 6k^2 \geq 0.$$

Also, we have

$$f_6(0,y,z) = (y-z)^4[(y+z)^2 + (2-k)yz] \geq 0.$$

□

Proof of Proposition 2.5. Write the inequality as $f_6(x,y,z) \geq 0$, where

$$f_6(x,y,z) = (xy+yz+zx)\sum(x-y)(x-z)(x-ky)(x-kz) - \alpha_k(x-y)^2(y-z)^2(z-x)^2.$$

We have

$$A = 27\alpha_k > 0,$$
$$f_6(x,1,1) = (2x+1)(x-1)^2(x-k)^2,$$

$$\begin{aligned} f_6(0,y,z) &= yz[(y^2+z^2)^2 - (k+1+\alpha_k)yz(y^2+z^2) + (k^2+2k-2+2\alpha_k)y^2z^2] \\ &= yz[(y+z)^2 - (4+|k|)yz]^2. \end{aligned}$$

To prove the desired inequality, we apply Theorem 2.2 for

$$\alpha = |k|, \quad \beta = -2, \quad \gamma = \frac{|k|}{2}, \quad \delta = \infty,$$

when

$$f_{|k|,-2}(x) = \frac{4(x-1)^4(x-|k|)^2}{81(2+|k|)^2},$$

$$f_{|k|/2,\infty}(x) = \frac{(x-1)^4(2x-|k|)^2}{9(1+|k|)^2(x+2)^2}, \quad \bar{f}_{|k|/2,\infty}(y,z) = \frac{y^2z^2[(y+z)^2-(4+|k|)yz]^2}{9(1+|k|)^2(y+z)^2}.$$

From

$$f_6(x,1,1) - Af_{|k|,-2}(x) = \frac{(x-1)^2[3(2+|k|)^2(2x+1)(x-k)^2 - 4\alpha_k(x-1)^2(x-|k|)^2]}{3(2+|k|)^2},$$

it follows that the condition (a) in Theorem 2.2 is satisfied if

$$3(2+|k|)^2(2x+1)(x-k)^2 \geq 4\alpha_k(x-1)^2(x-|k|)^2$$

for $0 \leq x \leq 4$. Since $(x-k)^2 \geq (x-|k|)^2$ and $2x+1 \geq (x-1)^2$, it suffices to show that

$$3(2+|k|)^2 \geq 4\alpha_k.$$

Indeed, for $k \leq 0$ we have

$$3(2+|k|)^2 - 4\alpha_k = 3k^2 \geq 0,$$

and for $k \geq 0$ we have

$$3(2+|k|)^2 - 4\alpha_k = k(3k+8) \geq 0.$$

The first condition in (b), namely $f_6(x,1,1) \geq Af_{|k|/2,\infty}(x)$ for $x > 4$, holds if

$$(1+|k|)^2(2x+1)(x+2)^2(x-k)^2 \geq 3\alpha_k(x-1)^2(2x-|k|)^2. \quad (5.3)$$

The second condition in (b), namely $f_6(0,y,z) \geq A\bar{f}_{|k|/2,\infty}(y,z)$ for $y,z \geq 0$, is equivalent to

$$yz[(y+z)^2 - (4+|k|)yz]^2[(1+|k|)^2(y+z)^2 - 3\alpha_k yz] \geq 0. \quad (5.4)$$

Case 1: $k \leq -5/4$. The inequality (5.3) is equivalent to

$$(1-k)(2x+1)(x+2)^2(x-k)^2 \geq 9(x-1)^2(2x+k)^2.$$

Since $4(x-k)^2 > (2x+k)^2$ and $2x+1 > 2(x-1)$, it suffices to show that

$$2(1-k)(x+2)^2 \geq 36(x-1).$$

Indeed,

$$2(1-k)(x+2)^2 - 36(x-1) > 3(x+2)^2 - 36(x-1) = 3(x-4)^2 \geq 0.$$

The inequality (5.4) is equivalent to

$$(1-k)yz[(y+z)^2 - (4-k)yz]^2[(1-k)(y+z)^2 - 9yz] \geq 0,$$

and is true for any $k \leq -5/4$ since

$$(1-k)(y+z)^2 - 9yz \geq 4(1-k)yz - 9yz = (-4k-5)yz \geq 0.$$

Case 2: $k \in \{1,2,3\}$. The inequality (5.4) is equivalent to

$$yz[(y+z)^2 - (k+4)yz]^2[(k+1)^2(y+z)^2 - 3(k+3)yz] \geq 0,$$

and is true since

$$(k+1)^2(y+z)^2 - 3(k+3)yz \geq [4(k+1)^2 - 3(k+3)]yz = (4k^2+5k-5)yz \geq 0.$$

As for the inequality (5.3), it holds if

$$(k+1)^2(2x+1)(x+2)^2(x-k)^2 \geq 3(k+3)(x-1)^2(2x-k)^2. \quad (5.5)$$

For $k=1$, the inequality (5.5) is true if

$$(2x+1)(x+2)^2 \geq 3(2x-1)^2.$$

Indeed,

$$(2x+1)(x+2)^2 - 3(2x-1)^2 > 6(2x+1)(x+2) - 3(2x-1)^2 = 3(14x+3) > 0.$$

For $k=2$, the inequality (5.5) has the form

$$3(2x+1)(x^2-4)^2 \geq 20(x-1)^4,$$

and is true since $x^2-4 > (x-1)^2$ and $3(2x+1) > 20$.

For $k=3$, the inequality (5.5) becomes

$$8(2x+1)(x+2)^2(x-3)^2 \geq 9(x-1)^2(2x-3)^2.$$

Since $8(2x+1) > 72 > 64$, it suffices to show that

$$8(x+2)(x-3) \geq 3(x-1)(2x-3).$$

Indeed,

$$8(x+2)(x-3)-3(x-1)(2x-3)=2x^2+7x-57>32+28-57>0.$$

Case 3: $k=0$. To prove the original inequality we can use Theorem 2.3 for $\alpha=0$, $\beta=-2$ and $\delta=0$. The condition (a) in Theorem 2.3 is the same as the condition (a) in Theorem 2.2. Thus, we only need to prove that $f_6(x,1,1)\geq Ax^2$ and $f_6(0,y,z)\geq 0$ for $x>4$, $y,z\geq 0$. Indeed,

$$f_6(x,1,1)-Ax^2=x^2[(2x+1)(x-1)^2-81]>0, \quad f_6(0,y,z)=yz(y-z)^4\geq 0.$$

□

Proof of Proposition 2.6. Write the inequality as $f_6(x,y,z)\geq 0$, where

$$f_6(x,y,z)=p\sum(y+z)(x-y)(x-z)(x-ky)(x-kz)-(2+|k|)^2(x-y)^2(y-z)^2(z-x)^2.$$

The polynomial $f_6(x,y,z)$ has the highest coefficient

$$A=27(2+|k|)^2.$$

Also, we have

$$f_6(x,1,1)=2(x+2)(x-1)^2(x-k)^2,$$
$$f_6(0,y,z)=yz[(y+z)^2-2(2+|k|)yz]^2.$$

We will apply Theorem 2.2 for

$$\alpha=|k|, \quad \beta=-2, \quad \gamma=|k|, \quad \delta=\infty,$$

when

$$f_{|k|,-2}(x)=\frac{4(x-1)^4(x-|k|)^2}{81(2+|k|)^2},$$

$$f_{|k|,\infty}(x)=\frac{4(x-1)^4(x-|k|)^2}{9(1+2|k|)^2(x+2)^2}, \quad \bar{f}_{|k|,\infty}(y,z)=\frac{y^2z^2[(y+z)^2-2(2+|k|)yz]^2}{9(1+2|k|)^2(y+z)^2}.$$

Since

$$f_6(x,1,1)-Af_{|k|,-2}(x)=\frac{(x-1)^2[3(x+2)(x-k)^2-2(x-1)^2(x-|k|)^2]}{3},$$

the condition (a) in Theorem 2.2 is fulfilled if

$$3(x+2)(x-k)^2\geq 2(x-1)^2(x-|k|)^2$$

for $0\leq x\leq 4$. This is true since $3(x+2)\geq 2(x-1)^2$ and $(x-k)^2\geq(x-|k|)^2$. Indeed,

$$3(x+2)-2(x-1)^2=(4-x)(1+2x)\geq 0,$$

$$(x-k)^2-(x-|k|)^2=2(|k|-k)x\geq 0.$$

The condition (b) is fulfilled if $f_6(x,1,1)\geq Af_{|k|,\infty}(x)$ and $f_6(0,y,z)\geq A\bar{f}_{|k|,\infty}(y,z)$ for $x>4$, $y\geq 0$, $z\geq 0$. The first condition holds if

$$(1+2|k|)^2(x+2)^3(x-k)^2\geq 6(2+|k|)^2(x-1)^2(x-|k|)^2.$$

Since $(x-k)^2\geq(x-|k|)^2$, it suffices to show that

$$(1+2|k|)^2(x+2)^3\geq 6(2+|k|)^2(x-1)^2.$$

Since $x+2>6$, it suffices to show that

$$(1+2|k|)^2(x+2)^2\geq(2+|k|)^2(x-1)^2,$$

which is equivalent to

$$(1+2|k|)(x+2) \geq (2+|k|)(x-1).$$

This is true for $x > 4$ and $|k| \geq 1$ because $1+2|k| \geq 2+|k|$ and $x+2 > x-1$.

The second condition in (b) is equivalent to

$$yz[(y+z)^2 - 2(2+|k|)yz]^2[(1+2|k|)^2)(y+z)^2 - 3(2+|k|)^2yz] \geq 0,$$

and is true if

$$4(1+2|k|)^2 - 3(2+|k|)^2 \geq 0,$$

or, equivalently,

$$2(1+2|k|) \geq \sqrt{3}(2+|k|).$$

Indeed,

$$2(1+2|k|) - \sqrt{3}(2+|k|) = (4-\sqrt{3})|k| + 2(1-\sqrt{3}) \geq (4-\sqrt{3}) + 2(1-\sqrt{3}) > 0.$$

6 Conclusion

In this paper, we have investigated the inequality $f_6(x,y,z) \geq 0$ for all nonnegative real numbers x,y,z, where $f_6(x,y,z)$ is a symmetric homogeneous polynomial of degree six. To prove an inequality of this type using the necessary and sufficient conditions from Theorem 1.2, we need to write the polynomial $f_6(x,y,z)$ in the form (1.2), which is a tedious and laborious work. Consequently, we have formulated and proved two new theorems with strong sufficient conditions to have $f_6(x,y,z) \geq 0$ for all nonnegative real numbers x,y,z. Six elaborate applications are given to show the effectiveness of the proposed sufficient conditions.

Acknowledgment

The author is thankful to the anonymous referees for their valuable comments.

Competing Interests

The author declares that no competing interests exist.

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