British Journal of Mathematics & Computer Science
4(5): 719-734, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

On the Superstability of a Generalization of the Cosine Equation

D. Zeglami[1*], S. Kabbaj[1] and A. Roukbi[1]

[1] *Department of Mathematics, Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco.*

Original Research Article

Received: 24 August 2013
Accepted: 24 October 2013
Published: 09 December 2013

Abstract

The aim of this paper is to investigate the stability problem for the functional equation:

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \qquad (E_{g,f})$$

and the superstability of the d'Alembert's equation:

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \qquad (A)$$

under the conditions from which the differences of each equation are bounded by $\varphi(x)$, $\psi(x)$ and $\min(\varphi(x), \psi(y))$ where G is an arbitrary group, not necessarily abelian, f, g are complex valued functions, φ, ψ are real valued functions and σ is an involution of G.

Keywords: Hyers-Ulam stability, Superstability, d'Alembert equation, Wilson's functional equation.

2000 Mathematics Subject Classification. Primary 39B72

1 Introduction

There is a strong stability phenomenon which is known as a superstability. An equation of homomorphism is called superstable if each approximate homomorphism is actually a true homomorphism. This property was first observed by J. Baker et al. [1] in the following Theorem:

Let V be a vector space. If a function $f : V \to IR$ satisfies the inequality

$$|f(x+y) - f(x)f(y)| \le \varepsilon,$$

for some $\varepsilon > 0$ and for all $x, y \in V$. Then either f is a bounded function or

Corresponding author: zeglamidriss@yahoo.fr;

$$f(x+y) = f(x)f(y) \quad \textit{for all } x, y \in V.$$

In light of this result, the stability of a class of functional equations has been investigated by Badora, Baker, Dragomir, Gàvruta, Ger, Kabbaj, Kim, Rassias, Roukbi, Tyrala, Székelyhidi, Zeglami etc.

In [2], R. Badora and R. Ger have improved the superstability problem of the classical d'Alembert's functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G, \qquad (C)$$

under the condition

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \ \ or \ \ \varphi(y).$$

Namely, the following theorem holds true.

Theorem 1. (R. Badora, R. Ger [2]) *Let* $(G,+)$ *be an Abelian group,* $f : G \to C$ *and let* $\varphi : G \to R$ *satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x) \, or \, \psi(y) \quad \textit{for all } x, y \in G.$$

Then either f *is bounded or* f *satisfies the classical d'Alembert's equation* (C).

In [3] A. Roukbi, D. Zeglami and S. Kabbaj proved the superstability of the eqaution

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \qquad (E_{f,g})$$

without imposing any conditions on the group G. Equation $(E_{f,g})$ is called the Wilson functional equation (see [4]) and sometimes, the first generalization of the d'Alembert's functional equation.

In the present paper, we consider, in both abelian and non abelian groups and without any conditions on f, the stability problem of the functional equation

$$f(xy) + f(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G, \qquad (E_{g,f})$$

under the condition

$|f(xy)+f(x\sigma(y))-2g(x)f(y)|\leq \varphi(x),\psi(y)\ or\ \min(\varphi(x),\psi(y))$ where G is any one group and σ is an involution of G, i. e. $\sigma(\sigma(x))=x$ and $\sigma(xy)=\sigma(y)\sigma(x)$ for all $x,y\in G$. The equation $(E_{g,f})$ is called, sometimes, second generalization of the cosine equation. As a consequence, we obtain the superstability of the d'Alembert's functional equation

$$f(xy)+f(x\sigma(y))=2f(x)f(y),\quad x,y\in G, \qquad (A)$$

which proved by Roukbi, Zeglami and Kabbaj $[3,5]$ on any group, by Redouani, Elqorachi and Rassias $[6]$ on step 2 nilpotent groups and by Baker, Badora and Ger, Gàvruta, Kim, etc ($[7]$, $[8]$, $[9]$, $[10]$, ...) in the case where G is an abelian group.
The interested reader should refer to $[1-3,4-25]$ for a thorough account on the subject of stability of functional equations.

In this paper, let G *be any one group,* e *denote its neutral element,* C *the field of complex numbers and* R *the field of real numbers. We may assume that* f *and* g *are complex valued functions on* G, $\varphi,\psi:G\to R$ *are mappings,* λ,δ *are nonnegative real constants, and* σ *is an involution of* G i. e. $\sigma(\sigma(x))=x$ and $\sigma(xy)=\sigma(y)\sigma(x)$ for all $x,y\in G$. *In the case that* $f(e)\neq 0$ *we put* $\widetilde{f}=\dfrac{1}{f(e)}f$.

A typical example of the involution σ *is the group involution* $\sigma(x)=x^{-1},\ x\in G$. *Another is the adjoint* $A\to A^*$ *in the matrix group* $GL(n,C)$ of $n\times n$ invertible matrices, *A third one is*

$$\sigma\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & y & z\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix}$$

on the Heisenberg group $H_3:=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}/\ x,y,z\in R\right\}$.

2. Solutions of the Equation $(E_{g,f})$

We start with solutions of the d'Alembert's functional equation: In 2008, Th. Davison $[26]$ proved the following result:

Lemma 1. *Let* G *be a topological group and* $f: G \to C$ *a continuous function with* $f(e) = 1$ *satisfying*

$$f(xy) + f(xy^{-1}) = 2f(x)f(y), \quad x, y \in G.$$

Then there is a continuous (group) homomorphism $h: G \to SL(2, C)$ *such that*

$$f(x) = \frac{1}{2} tr(h(x)) \text{ for all } x, y \in G.$$

Giving solutions of equation (A) the theory of representations is introduced by H. Stetkær in [27] . Precisely, he proved that:

Lemma 2. *Let* S *be a semigroup. The non-zero continuous solutions* f *of* (A) *on* S *are the functions of the form*

$$f(x) = \frac{1}{2} tr(\pi(x)), \; x \in G \tag{2.1}$$

where π ranges over the 2-dimensional continuous *representations of* S *for which*

$$\pi(\sigma(x)) = adj(\pi(x)) \tag{2.2}$$

for all $x \in S$ *and* $adj: Mat_2(C) \to Mat_2(C), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$

Note that the equation (A) is raised by Kannappan in the case where G *is abelian* [28]. Using Lemma 2 we directly prove the following fact concerning solutions of equation $(E_{g,f})$.

Theorem 2. *Let* G *be any group. Then* $f, g: G \to C$ *satisfy the equation* $(E_{g,f})$ *if and only if*

i) $f = 0$ *and* g *is arbitrary, or*

ii) $f \neq 0$ *and* $f(x) = \alpha g(x)$ *for all* $x \in G$*, where* $\alpha \in C - \{0\}$ *and* g *is a solution of* (A)

Furthermore, the non-zero continuous solutions f, g *of* $(E_{g,f})$ *on* G *are functions of the form* : $g = \frac{1}{2}\chi_\pi$; $f = \frac{\alpha}{2}\chi_\pi$ *where* $\alpha \in C - \{0\}$ *and* π *ranges over the 2-dimensional continuous representations of* G *satisfying (2.2).*

Proof. Assume that $f \neq 0$. Setting $y = e$ in $(E_{g,f})$ we have $f(x) = g(x)f(0)$ for all $x \in G$. From which we conclude that $f(0) \neq 0$. Putting $\alpha := f(0)$ we get that $f(x) = \alpha g(x)$ *for all* $x \in G$. So, from $(E_{g,f})$ we obtain

$$\alpha g(xy) + \alpha g(x\sigma(y)) = 2g(x)\alpha g(y), \quad x, y \in G,$$

for all $x, y \in G$. Then g is a solution of (A) and $f = \alpha g$. The rest of the proof follows from Lemma 2.

3. Stability of the Equation $(E_{g,f})$

Lemma 3. *Assume that functions* $f, g : G \to C$ *and* $\psi : G \to R$ *satisfy the inequality*

$$\left|f(xy) + f(x\sigma(y)) - 2g(x)f(y)\right| \leq \psi(y) \text{ , for all } x, y \in G \tag{3.1}$$

such that $f \neq 0$. *Then* f *is unbounded if and only if* g *is unbounded too.*

Proof. If $f(e) = 0$. Putting $y = e$ in (3.1) we get $\left|f(x)\right| \leq \frac{\psi(e)}{2}$, for all $x \in G$ i.e. f is bounded. Let $M = \sup\left|f\right|$ and choose $a \in G$ such that $f(a) \neq 0$ then we get from the inequality (3.1) that $\left|g(x)\right| \leq \frac{1}{2f(a)}(2M + \psi(a))$ for all $x \in G$, i.e. g is bounded too.

If $f(e)$ *is a non zero complex number,* substituting y by e in (3.1) we obtain

$$\left|f(x) - f(e)g(x)\right| \leq \frac{\psi(e)}{2},$$

for all $x \in G$, which shows that f is unbounded is equivalent to g is unbounded too.

Lemma 4. *Assume that functions* $f, g : G \to C$ *and* $\psi : G \to R$ *satisfy the inequality*

$\left|f(xy) + f(x\sigma(y)) - 2g(x)f(y)\right| \leq \psi(y)$, *for all* $x, y \in G$

Such that $f(e) = 1$. *Then*

i) $\left|g(xy) + g(x\sigma(y)) - 2g(x)f(y)\right| \leq \psi(y) + \psi(e)$, *for all* $x, y \in G$. (3.2)

ii) f *is unbounded if and only if* g *is also unbounded.*

Proof. i) Assume that $f(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show that

$$\left|f(x) - g(x)\right| \leq \frac{\psi(e)}{2} \tag{3.3}$$

for all $x \in G$. Let $F(x) := f(x) - g(x)$. By virtue of inequality (3.3), we have

$$g(x) = f(x) - F(x) \ \ and \ \ |F(x)| \leq \frac{\psi(e)}{2} \ \ , \tag{3.4}$$

for all $x \in G$. By the definition of F and the use of (3.1) we have

$$|g(xy) + g(x\sigma(y)) - 2g(x)f(y)| = |f(xy) - F(xy) + f(x\sigma(y)) - F(x\sigma(y)) - 2g(x)f(y)|$$
$$\leq |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| + |F(xy)| + |F(x\sigma(y))|$$
$$\leq \psi(y) + \psi(e).$$

ii) Follows from (3.3) and it is also a particular case of Lemma 3.

Lemma 5. *Assume that functions* $f, g : G \to C$ *and* $\varphi : G \to R$ *satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2g(x)f(y)| \leq \varphi(x) \ , \tag{3.5}$$

for all $x, y \in G$ *such that* $f(e) = 1$. *Then*

$$|g(xy) + g(x\sigma(y)) - 2g(x)f(y)| \leq 2\varphi(x) \ , \textit{for all}\ x, y \in G . \tag{3.6}$$

Proof. i) Assume that $f(e) = 1$. Putting $y = e$ in the inequality (3.1). It is easy to show that

$$|f(x) - g(x)| \leq \frac{\varphi(x)}{2} \tag{3.7}$$

for all $x \in G$. Let $F(x) := f(x) - g(x)$. By virtue of inequality (3.7), we have

$$g(x) = f(x) - F(x) \ \ and \ \ |F(x)| \leq \frac{\varphi(x)}{2} \tag{3.8}$$

for all $x \in G$. Using (3.5) and (3.8) we get

$$|g(xy) + g(x\sigma(y)) - 2g(x)f(y)| = |f(xy) - F(xy) + f(x\sigma(y)) - F(x\sigma(y)) - 2g(x)f(y)|$$

$$\leq |f(xy) + f(x\sigma(y)) - 2g(x)f(y)| + |F(xy)| + |F(x\sigma(y))|$$
$$\leq 2\varphi(x)$$

Theorem 3. *Assume that functions* $f, g : G \to C$ *and* $\psi : G \to R$ *satisfy the inequality*

$$|f(xy)+f(x\sigma(y))-2g(x)f(y)|\le\psi(y)\ ,$$

for all $x,y\in G$ *such that* $f\ne 0$. *Then either* g *(or* f *) is bounded or*

$$\widetilde{f}(xy)+\widetilde{f}(x\sigma(y))+\widetilde{f}(yx)+\widetilde{f}(\sigma(y)x)=4\widetilde{f}(x)\widetilde{f}(y), \tag{3.9}$$

for all $x,y\in G$, *where* $\widetilde{f}=\dfrac{1}{f(e)}f$.

Proof. i) Assume that f,g satisfy the inequality (3.1) such that g is unbounded (which is equivalent - by lemma 3 - to f is also unbounded).

First case: We start with the following particular case $f(e)=1$. For all $x,y,z\in G$ we have

$$2|g(z)||f(xy)+f(x\sigma(y))+f(yx)+f(\sigma(y)x)-4f(x)f(y)|$$

$$=|2g(z)f(xy)+2g(z)f(x\sigma(y))+2g(z)f(yx)+2g(z)f(\sigma(y)x)-8g(z)f(x)f(y)|$$

$$\begin{aligned}
&\le|f(zxy)+f(z\sigma(y)\sigma(x))-2g(z)f(xy)|\\
&\quad+|f(zx\sigma(y))+f(zy\sigma(x))-2g(z)f(x\sigma(y))|\\
&\quad+|f(zyx)+f(z\sigma(x)\sigma(y))-2g(z)f(yx)|\\
&\quad+|f(z\sigma(y)x)+f(z\sigma(x)y)-2g(z)f(\sigma(y)x)|\\
&\quad+|f(zxy)+f(zx\sigma(y))-2g(zx)f(y)|\\
&\quad+|f(zyx)+f(zy\sigma(x))-2g(zy)f(x)|\\
&\quad+|f(z\sigma(y)x)+f(z\sigma(y)\sigma(x))-2g(z\sigma(y))f(x)|\\
&\quad+|f(z\sigma(x)y)+f(z\sigma(x)\sigma(y))-2g(z\sigma(x))f(y)|\\
&\quad+2|f(y)||g(zx)+g(z\sigma(x))-2g(z)f(x)|\\
&\quad+2|f(x)||g(zy)+g(z\sigma(y))-2g(z)f(y)|.
\end{aligned}$$

By virtue of inequalities (3.1) and (3.2), we have

$$\begin{aligned}
2|g(z)||f(xy)+f(x\sigma(y))+f(yx)+f(\sigma(y)x)-4f(x)f(y)|&\\
\le\psi(xy)+\psi(x\sigma(y))+\psi(yx)+\psi(\sigma(y))x)+2\psi(x)&\\
+2\psi(y)+2|f(y)|(\psi(x)+\psi(e))+2|f(x)|(\psi(y)+\psi(e))&
\end{aligned}\tag{3.10}$$

If we fix x, y, the right hand side of the above inequality is bounded function of z. Since g is unbounded, from (3.10), we conclude that $f\,(=\widetilde{f})$ is a solution of the equation (3.9), which ends the proof in this case.

General case: If f is a non-zero function such that $f(e)=0$ then g and f are bounded (Proof of Lemma 3). For the case that $f(e)$ is any one non-zero complex number, dividing the two sides of the inequality (3.1) by $\alpha = f(e)$ we find that

$$\left|\widetilde{f}(xy)+\widetilde{f}(x\sigma(y))-2g(x)\widetilde{f}(y)\right| \le \frac{\psi(y)}{|\alpha|}, \quad \textit{for all } x, y \in G,$$

where $\widetilde{f} = \frac{1}{\alpha} f$. We see that the pair $(\widetilde{f}, g)$ satisfies the inequality (3.1) with $\widetilde{f}(e)=1$ which shows, using the first case result, that either f (or g) is bounded or $\widetilde{f}$ satisfies the equation (3.9) which finished this proof.

As a consequence of Theorem 3, we have the following result on the superstability of the equation (A).

Corollary 1. *Assume that functions* $f : G \to C$ *and* $\psi : G \to R$ *satisfy the inequality*

$$\left|f(xy)+f(x\sigma(y))-2f(x)f(y)\right| \le \psi(y), \tag{3.11}$$

for all $x, y \in G$. *Then either* f *is bounded or* f *satisfies the d'Alembert's long functional equation* (3.9) *). Further, in the latter case, if* G *is abelian then* f *satisfies the equation* (A).

Proof. Assume that f is unbounded function satisfying (3.11). Putting $f = g$ in Theorem 3 we get that $\widetilde{f}$ is a solution of the equation (3.9). Substituting y by e in (3.11) we obtain $\left|f(x)(f(e)-1)\right| \le \frac{\psi(e)}{2}$ for all $x \in G$. This inequality shows that $f(e)=1$ because f is unbounded. So $f = \widetilde{f}$ is a solution of (3.9) and if G is abelian then f satisfies the equation (A).

In the following theorem the stability of the equation $(E_{g,f})$ will be investigated on any group. For $f = 0$ the pair (f, g) is a trivial solution of the equation $(E_{g,f})$.

Theorem 4. *Assume that functions* $f, g : G \to C$ *and* $\varphi, \psi : G \to R$ *satisfy the inequality*

$$|f(xy)+f(x\sigma(y))-2g(x)f(y)|\le\varphi(x)\ \text{ and }\ \psi(y), \tag{3.12}$$

for all $x,y\in G$ *such that* $f\neq 0$. *Then either* f *(or* g *) is bounded or the pair* (f,g) *satisfies the equation*

$$g(xy)+g(x\sigma(y))=2g(x)\widetilde{f}(y),\quad x,y\in G. \tag{3.13}$$

Furthermore in the latter case the function $\widetilde{f}$ *satisfies the equation (3.9).*

Proof. Assume that f,g satisfy inequality (3.12) *such that* $f\neq 0$. If $f(e)=0$, we have seen in Proof of Lemma 3 that f and g are bounded. Suppose that f *(or* g *)* is unbounded then we necessarily have $f(e)\neq 0$. That $\widetilde{f}$ satisfies (3.9) *is* proven in Theorem 3.

First case: We start with the case $f(e)=1$. For all $x,y,z\in G$ we have

$$\begin{aligned}
2|f(z)||g(xy)+g(x\sigma(y))-2g(x)f(y)|&\\
&=|2f(z)g(xy)+2f(z)g(x\sigma(y))-4f(z)g(x)f(y)|\\
&\le|f(xyz)+f(xy\sigma(z))-2g(xy)f(z)|\\
&+|f(x\sigma(y)z)+f(x\sigma(y)\sigma(z))-2g(x\sigma(y))f(z)|\\
&+|f(xyz)+f(x\sigma(z)\sigma(y))-2g(x)f(yz)|\\
&+|f(xy\sigma(z))+f(xz\sigma(y))-2g(x)f(y\sigma(z))|\\
&+|f(x\sigma(y)z)+f(x\sigma(z)y)-2g(x)f(\sigma(z)y)|\\
&+|f(x\sigma(y)\sigma(z))+f(xzy)-2g(x)f(zy)|\\
&+|f(x\sigma(z)y)+f(x\sigma(z)\sigma(y))-2g(x\sigma(z))f(y)|\\
&+|f(xzy)+f(xz\sigma(y))-2g(xz)f(y)|\\
&+|2g(x)\{f(yz)+f(y\sigma(z))+f(zy)+f(\sigma(z)y)-4f(y)f(z)\}|\\
&+2|f(y)||g(xz)+g(x\sigma(z))-2g(x)f(z)|
\end{aligned}$$

In virtue of inequalities (3.12) and (3.6), we obtain

$$\begin{aligned}
2|f(z)||g(xy)+g(x\sigma(y))-2g(x)f(y)|&\\
&\le\varphi(xy)+\varphi(x\sigma(y))+4\varphi(x)+2\psi(y)+4\varphi(x)|f(y)|\\
&+2|g(x)||f(yz)+f(y\sigma(z))+f(zy)+f(\sigma(z)y)-4f(y)f(z)|.
\end{aligned}$$

Since g is unbounded (which is equivalent to f is unbounded) then, according to Theorem 3, f is a solution of the equation (3.9). So we conclude that

$$2|f(z)||g(xy)+g(x\sigma(y))-2g(x)f(y)| \le \varphi(xy)+\varphi(x\sigma(y))+4\varphi(x)+2\psi(y)+4\varphi(x)|f(y)|. \quad (3.14)$$

Again the right hand side of (3.14) as a function of z is bounded for all fixed x, y. Since f is unbounded, from (3.14), we see that the pair (g, f) satisfies the equation:

$$g(xy)+g(x\sigma(y)) = 2g(x)f(y), \quad x, y \in G,.$$

***General case**:* Now we suppose that $f(e)$ is a nonzero complex number. Dividing the two sides of the inequality (3.12) by $\alpha = f(e)$ we find that

$$\left|\tilde{f}(xy)+\tilde{f}(x\sigma(y))-2g(x)\tilde{f}(y)\right| \le \frac{\varphi(x)}{|\alpha|} \;\; \textit{and} \;\; \frac{\psi(y)}{|\alpha|} \textit{ for all } x, y \in G,$$

where $\tilde{f} = \frac{1}{\alpha} f$. We see that the pair $(\tilde{f}, g)$ satisfies the inequality (3.12) with $\tilde{f}(e) = 1$ which shows, using the first case result, that either f (or g) is bounded or the pair (f, g) satisfies the equation (3.13) which finished this proof.

As another consequence of Theorem 4, we have the following result on the superstability of the equation (A) on any group which generalizes the Baker's result on the classical d'Alembert functional equation on an abelian group $[7, Theorem\ 5]$.

Corollary 2. $[3,5]$ *Let $\delta > 0$ be given. Assume that the function $f : G \to C$ satisfies the inequality*

$$|f(xy)+f(x\sigma(y))-2f(x)f(y)| \le \delta,$$

for all $x, y \in G$. Then either f is bounded or f is a solution of the equation (A). Further, in the latter case if f is continuous on G then it has the form (2.1).

Proof. Using similar techniques as in Proof of Corollary 1 we see that if f is unbounded then we have $f(e) = 1$ implying that $\tilde{f} = f$. The rest of the proof follows on putting $f = g$ in Theorem 4 (iii).

From above Theorems we get also the superstability of the equation $(E_{g,f})$ on two particular cases:

Corollary 3. *Let G be an Abelian group (or at least f is central), and let $f, g : G \to C$ and $\varphi, \psi : G \to R$ satisfy the inequality*

$$|f(xy)+f(x\sigma(y))-2g(x)f(y)| \le \varphi(x) \;\; \textit{and} \;\; \psi(y), \quad (3.15)$$

for all $x, y \in G$ *. Then there are the following possibilities:*

i) *If* $f = 0$*, then* g *is arbitrary.*
ii) *If* $g = 0$*, then* f *is bounded.*
iii) *If* $f \neq 0 \neq g$ *and* f *is bounded, then* g *is bounded, too.*
iv) *If* $g \neq 0$ *and* f *is unbounded, then* g *is unbounded, too. Moreover* g *is a solutions of* (A) *and the pair* (f, g) *satisfies equations* $(E_{g,f})$ *and* $(E_{g,f})$.

Proof. (ii) If $g = 0$ then the inequality (3.15) has a form $|f(xy) + f(x\sigma(y))| \leq \varphi(x)$ *and* $\psi(y)$ *for all* $x, y \in G$. *Put* $y = e$*, we get* $|f(x)| \leq \dfrac{\psi(e)}{2}$ *for all* $x \in G$ *i.e.* f *is bounded.*

(iii) If $f \neq 0 \neq g$ and f is bounded, let $M = \sup|f|$ and choose $a \in G$ such that $f(a) \neq 0$ then we get from the inequality (3.15) that $|g(x)| \leq \dfrac{1}{2|f(a)|}(2M + \psi(a))$ for all $x \in G$, i.e. g is bounded too.

To get (iv) we use Theorem 4 in which we have seen that if f is unbounded then $f(e) \neq 0$,

$g(xy) + g(x\sigma(y)) = 2g(x)\widetilde{f}(y), \quad x, y \in G$ *and*

$\widetilde{f}(xy) + \widetilde{f}(x\sigma(y)) + \widetilde{f}(yx) + \widetilde{f}(\sigma(y)x) = 4\widetilde{f}(x)\widetilde{f}(y)$, *for all* $x, y \in G$.

If G *is abelian or at least* f *is central (i.e.* $f(xy) = f(yx)$ *for all* $x, y \in G$ *) then we get*

$$\widetilde{f}(xy) + \widetilde{f}(x\sigma(y)) = 2\widetilde{f}(x)\widetilde{f}(y), \tag{3.16}$$

for all $x, y \in G$. *Dividing the two sides of the inequality* (3.15) *by* $\alpha = f(e)$ *we find that*

$$\left|\widetilde{f}(xy) + \widetilde{f}(x\sigma(y)) - 2g(x)\widetilde{f}(y)\right| \leq \frac{\varphi(x)}{|\alpha|} \text{ and } \frac{\psi(y)}{|\alpha|}, \tag{3.17}$$

for all $x, y \in G$. *When we substitute* (3.16) *into* (3.17) *we get that*

$$\left|2\widetilde{f}(y)(\widetilde{f}(x) - g(x))\right| \leq \frac{\varphi(x)}{|\alpha|} \text{ and } \frac{\psi(y)}{|\alpha|}, \tag{3.18}$$

for all $x, y \in G$. Since f is unbounded then so is $\widetilde{f}$. Consequently (3.18) implies $\widetilde{f} = g$. Thus g is a solution of (A). Substituting $\widetilde{f}$ by g on the second (resp. the last) Factor of the right hand side of (3.16) the expression reduces to $(E_{g,f})$ and $(E_{f,g})$.

Corollary 4. *Let G be any group, and let $f, g : G \to C$ satisfy the inequality (3.15) such that $g(\sigma(x)) = g(x)$ for all $x \in G$. Then if f is unbounded, then g is unbounded, too. Moreover g is a solutions of (A) and (f, g) satisfies equations $(E_{g,f})$ and $(E_{f,g})$.*

Proof. Suppose that f, g satisfy (3.15) with $g(\sigma(x)) = g(x)$ *for all* $x \in G$. *If* f is unbounded, using Theorem 4, we obtain the equality (3.13). By putting $y = e$ *in (3.13)* it is easy to see that $\tilde{f}(y) = \frac{1}{g(e)} g(y)$ (the case $g(e) = 0$ does not occur here due to our assumption that f is unbounded). Using this equality and (3.13) we get

$$\begin{aligned}
\tilde{f}(xy) + \tilde{f}(x\sigma(y)) &= \frac{1}{g(e)}(g(xy) + g(x\sigma(y))) \\
&= \frac{1}{g(e)}(2g(x)\tilde{f}(y)) \\
&= 2\tilde{f}(x)\tilde{f}(y),
\end{aligned}$$

and the rest of the proof runs along the same lines as in proof of Corollary 3 (iv*)*.

Remarks.

i) In the case where G is an abelian group and f, g satisfy the inequality (3.15) we know –according to Corollary 3- that if f is unbounded then g is a solutions of (A) but does not always f as shown by the example: Let $f, g : R \to R$ be functions with $g(x) = ch(x) := \frac{e^{ix} + e^{-ix}}{2}$ and $f(x) = 3ch(x)$ and let $\sigma(x) = -x$ for all $x \in R$. Then

$$|f(x+y) + f(x+\sigma(y)) - 2g(x)f(y)| = 0 \ ,$$

but f is unbounded and f does not satisfy the equation (A).

ii) Let $f, g : R \to R$ be functions with $f(x) = x^2 + 1$ and $g(x) = 1$ for all $x \in IR$, and let $\sigma(x) = -x$. Then

$$|f(x+y) + f(x+\sigma(y)) - 2g(x)f(y)| = 2x^2 = \varphi(x),$$

and $f(0) = 1$ but f is unbounded and f, g do not satisfy the equation

$$g(x+y) + g(x+\sigma(y)) = 2g(x)\tilde{f}(y). \qquad (3.19)$$

This shows that the condition

$$\left|f(xy)+f(x\sigma(y))-2g(x)f(y)\right|\leq\psi(y),\ \ x,y\in G \tag{3.20}$$

is essential in the case (iii) of Theorem 4. This example shows also that the condition (3.20) is essential in Theorem 3.

iii) Let $f,g:R\rightarrow R$ be functions with $f(x)=ch(x)$ and $g(x)=1+ch(x)$ for all $x\in R$, and let $\sigma(x)=-x$. Then

$$\left|f(x+y)+f(x+\sigma(y))-2g(x)f(y)\right|=2ch(y)=\psi(y),$$

and $f(0)=1$ but f is unbounded and f,g do not satisfy the equation (3.19). This shows that the condition

$$\left|f(xy)+f(x\sigma(y))-2g(x)f(y)\right|\leq\varphi(x),\quad x,y\in G,$$

is essential in the case (iii) of Theorem 4.

iv) The obtained results in this paper can be extended to the equation

$f(xy)+f(x\sigma(y))=\lambda g(x)f(y),\quad x,y\in G$, and λ is a complex constant.

It can be also extended to the commutative semi simple Banach algebra on any group as in $[10,17,18]$ in the case where G is an abelian group.

v) If we apply the combinaison of cases
(a) $g=f$ or $g\neq f$.
(b) $\sigma(x)$, $\sigma(x)=x$, or $\sigma(x)=-x$.
(c) $\varphi(x)=\psi(x)=\delta$ or $\varphi(x)=\psi(x)\neq\delta$.
(d) The group G is abelian or non abelian.

to Theorem 3 and Theorem 4, we obtain some results of the papers $[1-3,4-25]$.

Acknowledgements

We thank the referees very much for their helpful comments.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Baker JA, Lawrence J, Zorzitto F. The stability of the equation $f(x+y)=f(x)+f(y)$. Proc. Amer. Math. Soc. 1979;74:242-246.

[2] Badora R, Ger R. On some trigonometric functional inequalities, Functional Equations-Results and Advances, in: Adv. Math. 2002;3:3-15.

[3] Roukbi A, Zeglami D, Kabbaj S. Hyers-Ulam stability of Wilson's functional equation. Math. Sciences: Adv. and Appl. 2013;22:19-26.

[4] Stetkaer H. D'Alembert's and Wilson's functional equations on step 2 nilpotent groups. Aequationes Math. 2004;67(3):241-262.

[5] Zeglami D, Roukbi A, Kabbaj S. Hyers-Ulam Stability of Generalized Wilson's and d'Alembert's Functional equations, Afr. Mat. DOI: 10.1007/s13370-013-0199-6.

[6] Redouani A, Elqorachi E, Rassias Th.M. The superstability of d'Alembert's functional equation on step 2 nilpotent groups. Aeq. Math. 2007;74(3):237-248.

[7] Baker JA. The stability of the cosine equation. Proc. Amer. Math. Soc. 1980;80:411-416.

[8] Badora R. On Hyers-Ulam stability of Wilson's functional equation. Aequationes Math. 2000;60:211-218.

[9] Gàvruta P. On the stability of some functional equations, in: "Stability of mappings of Hyers-Ulam type", (Eds. Th. M. Rassias and J. Tabor), Hadronic Press. 1994;93-98.

[10] Kim GH. The Stability of the d'Alembert and Jensen type functional equations. Jour. Math. Anal & Appl. 2007;325:237-248.

[11] Bouikhalene B, Elqorachi E, Rassias JM. The superstability of d'Alembert's functional equation on the Heisenberg group, Appl. Math. Lett. 2010;23(1):105-109.

[12] Cholewa PW. The stability of the sine equation. Proceedings of the American Math. Society. 1983;88(4):631-634.

[13] Elqorachi E, Akkouchi M. On Hyers-Ulam stability of Cauchy and Wilson equations. Georgian Math. J. 2004;11(1):69-82.

[14] Hyers DH. On the stability of the linear functional equation. Proc. Nat. Acad. Sci. U.S.A. 1941;27:222-224.

[15] Hyers DH, Isac GI, Rassias Th.M. Stability of functional equations in several variables. Progress in Nonlinear Differential Equations and their Applications, 34. Birkhauser Boston, Inc., Boston, MA; 1998.

[16] Jung S. Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, New York; 2011.

[17] Kannappan Pl, Kim GH. On the stability of the generalized cosine functional equations. Ann. Acad. Pedagog. Crac. Stud. Math. 2001;1:49-58.

[18] Kim GH, Dragomir S. On the Stability of generalized d'Alembert and Jensen functional equation, Intern. Jour. Math. & Math. Sci. Vol. 2006 (2006), Article ID 43185.

[19] Rassias Th. M. On the stability of linear mapping in Banach spaces, Proc. Amer. Math. Soc., 1978;72:297-300.

[20] Rezaei H, Sharifzadeh M. On the super-stability of exponential Hilbertvalued functional equations. Journal of Inequalities and Applications; 2011. doi:10.1186/1029-242X-2011-114.

[21] Rezaei H. On the super-stability of trigonometric Hilbert-valued functional equations. Mathematical Inequalities Applications (In Press); (2012).

[22] Székelyhidi L. On a theorem of Baker. Lawrence and Zorzitto. Proc. Amer. Math. Soc. 1982;84:95-96.

[23] Székelyhidi L. On a stability theorem. C. R. Math. Rep. Acad. Sci. Canada. 1981;3:253-255.

[24] Tyrala I. The stability of the second generalization of d'Alembert's functional equation, Scientic Issues, Mathematics XIII, Częstochowa; 2008.

[25] ULAM SM. A Collection of Mathematical Problems, Interscience Publ. New York; 1961. Problems in Modern Mathematics, Wiley, New York; 1964.

[26] Davison TMK. D'Alembert's functional equation on topological monoids. Publ. Math. Debrecen. 2009;75(1-2):41-66.

[27] Kannappan Pl. The functional equation $f(xy)+f(xy^{-1})=2f(x)f(y)$ for groups. proc. Amer. Maths. Soc. 1968;19:69-74.

[28] Stetkær H. Functionals equations on groups. World Scientific Publishing Company; 2013.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=360&id=6&aid=2665