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# Generalized $n$-Tupled Common Fixed Point Theorems for Contractive Rational Type Condition

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## Abstract

Aims/ objectives: In this paper, we prove results on $n$-tupled coincidence point (for even $n$) for a pair of mappings without mixed monotone property satisfying a contractive condition of rational type in partially ordered metric spaces. Our main theorem improves the corresponding results of Chandok *et al.* (Int. Jour. of Math. Anal., Vol. 7, 2013, No. 9, 433-440).

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## 1 Introduction

In recent years, an extension of Banach's contraction principle has been considered by many authors in different metric spaces. It has fruitful applications within as well as outside mathematics. Generalizations of this principle continues to be an active area of research. Many authors have extended this theorem employing relatively more general contractive conditions ensuring the existence of a fixed point. The investigation of fixed points in ordered metric spaces is a relatively new development which appears to have its origin (in 2004) in the paper of Ran and Reurings [1] which was well complimented by the paper of Nieto and Lopez [2]. For similar other results in ordered metric spaces, one can be referred to ([1]-[23]).

In [3], Bhaskar and Lakshmikantham introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ wherein $(X, \preceq, d)$ be a partial metric space and also proved some coupled fixed point theorems in partially ordered complete metric spaces. In 2009, Bhaskar and Ćirić [4] proved coupled coincidence and coupled fixed point theorems for nonlinear contractive mappings in

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these spaces. Recently, Karapinar [5] introduced the concept of a quadruple fixed point and mixed monotone property of a mapping $F : X \times X \times X \times X \to X$ and obtained some quadruple fixed point theorems in partially ordered metric spaces. Extending this work, quadruple fixed point theorems are developed and related fixed point theorems are proved in ([5]-[11]).

Most recently, Imdad *et al.* [12] introduced the concepts of $n$-tupled coincidence as well as $n$-tupled fixed point and utilize these two definitions to obtain $n$-tupled coincidence as well as $n$-tupled common fixed point theorems for nonlinear mappings satisfying $\phi$-contraction condition in partially ordered complete metric spaces.

In [13], Doric *et al.* showed that a mixed monotone property in coupled fixed point results for mappings in ordered metric spaces can be replaced by another property which is often easy to check. In particular, it is automatically satisfied in the case of a totally ordered space. Hence these results can be applied in a much wider class of problems. The purpose of this paper is to present some $n$-tupled coincidence point results for a pair of mappings without mixed monotone property satisfying a contractive condition of rational type in metric spaces equipped with a partial ordering. Also we present results on the existence and uniqueness of $n$-tupled common fixed points.

# 2 Preliminaries

The following notions were introduced in [3].

Let $(X, \preceq)$ be a partially ordered set equipped with a metric $d$ such that $(X, d)$ is a metric space. We endow the product space $X \times X$ with the following partial ordering:

for $(x, y), (u, v) \in X \times X$, define $(u, v) \preceq (x, y) \Leftrightarrow x \succeq u, y \preceq v.$

Now we present some basic notions and results related to coupled fixed point in metric spaces.

**Definition 2.1.** Let $(X, \preceq)$ be a partially ordered set and $F : X \to X$ be a mapping. Then $F$ is said to be nondecreasing if for all $x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and nonincreasing if for all $x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$.

**Definition 2.2.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then $F$ and $g$ are said to be commute if $F(gx_1, gx_2) = g(F(x_1, x_2))$, for all $x_1, x_2 \in X$.

**Definition 2.3.** [4] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then $F$ is said to have mixed $g$-monotone property if for any $x, y \in X, F(x, y)$ is monotone $g$-nondecreasing in its first argument and monotone $g$-nonincreasing in its second argument, that is, for

$$x_1, x_2 \in X,\ gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$
$$y_1, y_2 \in X,\ gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

If $g = I$ (identity mapping) in Definition 2.3, then the mapping $F$ is said to have the mixed monotone property.

**Definition 2.4.** [14] Two mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{cases} \lim\limits_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \\ \lim\limits_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0, \end{cases}$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that

$$\begin{cases} \lim\limits_{n\to\infty} F(x_n, y_n) = \lim\limits_{n\to\infty} gx_n = x \\ \lim\limits_{n\to\infty} F(y_n, x_n) = \lim\limits_{n\to\infty} gy_n = y, \end{cases}$$

for some $x, y \in X$ are satisfied.

**Definition 2.5.** [4] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx \text{ and } F(y, x) = gy.$$

If $g = I$ (identity mapping) in Definition 2.5, then $(x, y) \in X \times X$ is called a coupled fixed point.

**Definition 2.6.** [5] An element $(x, y, z, w) \in X \times X \times X \times X$ is called a quadruple coincidence point of the mappings $F : X \times X \times X \times X \to X$ and $g : X \to X$ if

$$F(x, y, z, w) = gx,\ F(y, z, w, x) = gy,\ F(z, w, x, y) = gz \text{ and } F(w, x, y, z) = gw.$$

If $g = I$ (identity mapping) in Definition 2.6, then $(x, y, z, w) \in X \times X \times X \times X$ is called a quadruple fixed point.

If elements $x, y$ of a partially ordered set $(X, \preceq)$ are comparable (that is, $x \preceq y$ or $y \preceq x$ holds) we will write $x \lesseqqgtr y$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then we consider the following condition:
If $x, y, u, v \in X$ are such that $gx \lesseqqgtr F(x, y) = gu$, then $F(x, y) \lesseqqgtr F(u, v)$.
If $g$ is an identity mapping then for all $x, y, v$ if $x \lesseqqgtr F(x, y)$, then $F(x, y) \lesseqqgtr F(F(x, y), v)$.

**Theorem 2.1.** [15] *Let $(X, \preceq, d)$ be a complete partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. Suppose that the following hold:*
*(a) $g$ is continuous and $g(X)$ is closed;*
*(b) $F(X \times X) \subseteq g(X)$ and $g$ and $F$ are compatible;*
*(c) for all $x, y, u, v \in X$, if $gx \lesseqqgtr F(x, y) = gu$, then $F(x, y) \lesseqqgtr F(u, v)$,*
*(d) there exist $x_0, y_0 \in X$ such that $gx_0 \lesseqqgtr F(x_0, y_0)$ and $gy_0 \lesseqqgtr F(y_0, x_0)$,*
*(e) there exists $\alpha \in [0, 1)$ such that for all $x, y, u, v \in X$, with $gx \lesseqqgtr gu$ and $gy \lesseqqgtr gv$, satisfies,*
$d(F(x, y), F(u, v)) \leq \alpha \max \{d(gx, gu), d(gy, gv), \frac{d(gx,F(x,y))d(gu,F(u,v))}{d(gx,gu)}, \frac{d(gx,F(u,v))d(gu,F(x,y))}{d(gx,gu)}$

$$\frac{d(gy, F(y, x))d(gv, F(v, u))}{d(gy, gv)}, \frac{d(gy, F(v, u))d(gv, F(y, x))}{d(gy, gv)}\}, \tag{2.1}$$

*(f) $F$ is continuous.*
*Then there exist $x, y \in X$ such that $F(x, y) = gx$ and $F(y, x) = gy$, that is, $F$ and $g$ have a coupled coincidence point $(x, y) \in X \times X$.*

Throughout the paper, we consider $n$ to be an even positive integer. We begin with the following definitions (here $X^n = X \times X \times X \times ... \times X (n$ times)):

**Definition 2.7.** [12] Let $(X, \preceq)$ be a partially ordered set. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then the mapping $F$ is said to have the mixed $g$-monotone property if $F$ is $g$-nondecreasing in its odd position arguments and $g$-nonincreasing in its even position arguments, that is, if,
for all $x_1^1, x_2^1 \in X,\ gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n)$
for all $x_1^2, x_2^2 \in X,\ gx_1^2 \preceq gx_2^2 \Rightarrow F(x^1, x_1^2, x^3, ..., x^n) \succeq F(x^1, x_2^2, x^3, ..., x^n)$
for all $x_1^3, x_2^3 \in X,\ gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n)$

$$\vdots$$

for all $x_1^n, x_2^n \in X,\ gx_1^n \preceq gx_2^n \Rightarrow F(x^1, x^2, x^3, ..., x_1^n) \succeq F(x^1, x^2, x^3, ..., x_2^n)$.

If $g = I$ (identity mapping) in Definition 2.7, then the mapping $F$ is said to have the mixed monotone property.

**Definition 2.8.** [12] Let $(X, d)$ be a metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Then $F$ and $g$ are said to be commute if

$$F(gx^1, gx^2, ..., gx^n) = g(F(x^1, x^2, ..., x^n)) \text{ for all } x^1, x^2, ..., x^n \in X.$$

**Definition 2.9.** Two mappings $F : X^n \to X$ and $g : X \to X$ are said to be compatible if

$$\begin{cases} \lim_{m\to\infty} d(g(F(x_m^1, x_m^2, x_m^3, ..., x_m^n)), F(gx_m^1, gx_m^2, gx_m^3, ..., gx_m^n)) = 0 \\ \lim_{m\to\infty} d(g(F(x_m^2, x_m^3, ..., x_m^n, x_m^1)), F(gx_m^2, gx_m^3, ..., gx_m^n, gx_m^1)) = 0 \\ \vdots \\ \lim_{m\to\infty} d(g(F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1})), F(gx_m^n, gx_m^1, gx_m^2, ..., gx_m^{n-1})) = 0, \end{cases}$$

where $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ are sequences in $X$ such that

$$\begin{cases} \lim_{m\to\infty} F(x_m^1, x_m^2, x_m^3, ..., x_m^n) = \lim_{m\to\infty} gx_m^1 = x^1 \\ \lim_{m\to\infty} F(x_m^2, x_m^3, ..., x_m^n, x_m^1) = \lim_{m\to\infty} gx_m^2 = x^2 \\ \vdots \\ \lim_{m\to\infty} F(x_m^n, x_m^1, x_m^2, ..., x_m^{n-1}) = \lim_{m\to\infty} gx_m^n = x^n, \end{cases}$$

for some $x^1, x^2, ..., x^n \in X$ are satisfied.

**Definition 2.10.** [12] An element $(x^1, x^2, ..., x^n) \in X^n$ is called an $n$-tupled coincidence point of $F : X^n \to X$ and $g : X \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, ..., x^n) = gx^1 \\ F(x^2, x^3, ..., x^n, x^1) = gx^2 \\ F(x^3, ..., x^n, x^1, x^2) = gx^3 \\ \vdots \\ F(x^n, x^1, x^2, ..., x^{n-1}) = gx^n. \end{cases}$$

If $g = I$ (identity mapping) in Definition 2.10, then $(x^1, x^2, ..., x^n) \in X^n$ is called an $n$-tupled fixed point.

**Remark 2.1.** Definition 2.10 with $n = 2, 4$ respectively yield the definitions of coupled coincidence point and quadrupled coincidence point.

# 3 Main Results

Now our main result is as follows:

**Theorem 3.1.** *Let $(X, \preceq, d)$ be a complete partially ordered metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Suppose that the following hold:*
*(a) $g$ is continuous and $g(X)$ is closed;*
*(b) $F(X^n) \subseteq g(X)$ and $g$ and $F$ are compatible;*
*(c) if $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$, are such that $gx^1 \lesseqgtr F(x^1, x^2, ..., x^n) = gy^1$, then $F(x^1, x^2, ..., x^n) \lesseqgtr F(y^1, y^2, ..., y^n)$,*
*(d) there exist $x_0^1, x_0^2, ..., x_0^n \in X$ such that*

$$gx_0^1 \lesseqgtr F(x_0^1, x_0^2, ..., x_0^n), gx_0^2 \lesseqgtr F(x_0^2, ..., x_0^n, x_0^1), ..., gx_0^n \lesseqgtr F(x_0^n, x_0^1, ..., x_0^{n-1}),$$

*(e) there exists* $\alpha \in [0,1)$ *such that for all* $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$, *for which* $gx^1 \lesseqqgtr gy^1,\ gx^2 \lesseqqgtr gy^2, ..., gx^n \lesseqqgtr gy^n$, *with* $gx^1 \neq gy^1, gx^2 \neq gy^2, ..., gx^n \neq gy^n$, *satisfies,*

$$\begin{aligned} d(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \leq \alpha \max \{ & d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n), \\ & \frac{d(gx^1, F(x^1, x^2, ..., x^n))d(gy^1, F(y^1, y^2, ..., y^n))}{d(gx^1, gy^1)}, \\ & \frac{d(gx^2, F(x^2, ..., x^n, x^1))d(gy^2, F(y^2, ..., y^n, y^1))}{d(gx^2, gy^2)}, ..., \\ & \frac{d(gx^n, F(x^n, x^1, ..., x^{n-1}))d(gy^n, F(y^n, y^1, ..., y^{n-1}))}{d(gx^n, gy^n)}, \\ & \frac{d(gx^1, F(y^1, y^2, ..., y^n))d(gy^1, F(x^1, x^2, ..., x^n))}{d(gx^1, gy^1)}, \\ & \frac{d(gx^2, F(y^2, ..., y^n, y^1))d(gy^2, F(x^2, ..., x^n, x^1))}{d(gx^2, gy^2)}, ..., \\ & \frac{d(gx^n, F(y^n, y^1, ..., y^{n-1}))d(gy^n, F(x^n, x^1, ..., x^{n-1}))}{d(gx^n, gy^n)} \}, \end{aligned} \tag{3.1}$$

*(f)* $F$ *is continuous.*
*Then there exist* $x^1, x^2, ..., x^n \in X$ *such that* $F(x^1, x^2, ..., x^n) = gx^1,\ F(x^2, ..., x^n, x^1) = gx^2, ..., F(x^n, x^1, ..., x^{n-1}) = gx^n$, *that is,* $F$ *and* $g$ *have an* $n$*-tupled coincidence point* $(x^1, x^2, ..., x^n) \in X^n$.

**Proof.** Using conditions (b) and (d), construct sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$ in $X$ satisfying

$$\begin{cases} gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, x_{m-1}^3, ..., x_{m-1}^n) \\ gx_m^2 = F(x_{m-1}^2, x_{m-1}^3, ..., x_{m-1}^n, x_{m-1}^1) \\ \vdots \\ gx_m^n = F(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^{n-1}) \text{ for } m \geq 1. \end{cases}$$

By (d), $gx_0^1 \lesseqqgtr F(x_0^1, x_0^2, ..., x_0^n) = gx_1^1$ and condition (c) implies that

$$gx_1^1 = F(x_0^1, x_0^2, ..., x_0^n) \lesseqqgtr F(x_1^1, x_1^2, ..., x_1^n) = gx_2^1.$$

Proceeding by induction, we get that $gx_{m-1}^1 \lesseqqgtr gx_m^1$ and similarly

$$gx_{m-1}^2 \lesseqqgtr gx_m^2,\ gx_{m-1}^3 \lesseqqgtr gx_m^3, ..., gx_{m-1}^n \lesseqqgtr gx_m^n \text{ for each } m \geq 1.$$

Now from contractive condition (3.1), we have

$$\begin{aligned} d(gx_{m+1}^1, gx_m^1) &= d(F(x_m^1, x_m^2, ..., x_m^n), F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n)) \\ &\leq \alpha \max \Big\{ d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n), \\ & \frac{d(gx_m^1, F(x_m^1, x_m^2, ..., x_m^n))d(gx_{m-1}^1, F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n))}{d(gx_m^1, gx_{m-1}^1)}, \\ & \frac{d(gx_m^2, F(x_m^2, ..., x_m^n, x_m^1))d(gx_{m-1}^2, F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1))}{d(gx_m^2, gx_{m-1}^2)}, ..., \\ & \frac{d(gx_m^n, F(x_m^n, x_m^1, ..., x_m^{n-1}))d(gx_{m-1}^n, F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}))}{d(gx_m^n, gx_{m-1}^n)}, \end{aligned}$$

$$\frac{d(gx_m^1, F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n))d(gx_{m-1}^1, F(x_m^1, x_m^2, ..., x_m^n))}{d(gx_m^1, gx_{m-1}^1)},$$
$$\frac{d(gx_m^2, F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1))d(gx_{m-1}^2, F(x_m^2, ..., x_m^n, x_m^1))}{d(gx_m^2, gx_{m-1}^2)}, ...,$$
$$\left.\frac{d(gx_m^n, F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}))d(gx_{m-1}^n, F(x_m^n, x_m^1, ..., x_m^{n-1}))}{d(gx_m^n, gx_{m-1}^n)}\right\}$$
$$= \alpha \max \left\{ d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n), \right.$$
$$\frac{d(gx_m^1, gx_{m+1}^1)d(gx_{m-1}^1, gx_m^1)}{d(gx_m^1, gx_{m-1}^1)}, \frac{d(gx_m^2, gx_{m+1}^2)d(gx_{m-1}^2, gx_m^2)}{d(gx_m^2, gx_{m-1}^2)}, ...,$$
$$\frac{d(gx_m^n, gx_{m+1}^n)d(gx_{m-1}^n, gx_m^n)}{d(gx_m^n, gx_{m-1}^n)}, \frac{d(gx_m^1, gx_m^1)d(gx_{m-1}^1, gx_{m+1}^1)}{d(gx_m^1, gx_{m-1}^1)}$$
$$\left.\frac{d(gx_m^2, gx_m^2)d(gx_{m-1}^2, gx_{m+1}^2)}{d(gx_m^2, gx_{m-1}^2)}, ..., \frac{d(gx_m^n, gx_m^n)d(gx_{m-1}^n, gx_{m+1}^n)}{d(gx_m^n, gx_{m-1}^n)}\right\}$$
$$= \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),$$
$$d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\}. \tag{3.2}$$

Similarly we have,

$$d(gx_{m+1}^2, gx_m^2) \leq \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),$$
$$d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\}.$$
$$\vdots$$
$$d(gx_{m+1}^n, gx_m^n) \leq \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n),$$
$$d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\}.$$

Let

$$\sigma_m = \max \{d(gx_{m+1}^1, gx_m^1), d(gx_{m+1}^2, gx_m^2), ..., d(gx_{m+1}^n, gx_m^n)\}.$$

Hence

$$\max \{d(gx_{m+1}^1, gx_m^1), d(gx_{m+1}^2, gx_m^2), ..., d(gx_{m+1}^n, gx_m^n)\}$$
$$\leq \alpha \max \{d(gx_m^1, gx_{m-1}^1), d(gx_m^2, gx_{m-1}^2), ..., d(gx_m^n, gx_{m-1}^n)\} = \alpha\sigma_{m-1}.$$

By induction we get that

$$\max\{d(gx_m^1, gx_{m+1}^1), d(gx_m^2, gx_{m+1}^2), ..., d(gx_m^n, gx_{m+1}^n)\} \leq \alpha^m \sigma_0.$$

It easily follows that for each $m, l \in \mathbb{N}$ with $l < m$ we have

$$d(gx_l^1, gx_m^1) \leq \frac{\alpha^l}{1-\alpha}\sigma_0,\ d(gx_l^2, gx_m^2) \leq \frac{\alpha^l}{1-\alpha}\sigma_0, ..., d(gx_l^n, gx_m^n) \leq \frac{\alpha^l}{1-\alpha}\sigma_0.$$

Therefore $\{gx_m^1\}, \{gx_m^2\}, ..., \{gx_m^n\}$ are Cauchy sequences and since $g(X)$ is closed in a complete metric space, there exist $x^1, x^2, ..., x^n \in g(X)$ such that

$$\lim_{m\to\infty} gx_m^1 = \lim_{m\to\infty} F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n) = x^1$$
$$\lim_{m\to\infty} gx_m^2 = \lim_{m\to\infty} F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1) = x^2$$
$$\vdots$$

$$\lim_{m\to\infty} gx_m^n = \lim_{m\to\infty} F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}) = x^n.$$

Compatibility of $F$ and $g$ implies that

$$\lim_{m\to\infty} d(g(F(x_m^1, x_m^2, ..., x_m^n)), F(gx_m^1, gx_m^2, ..., gx_m^n)) = 0$$

$$\lim_{m\to\infty} d(g(F(x_m^2, ..., x_m^n, x_m^1)), F(gx_m^2, ..., gx_m^n, gx_m^1)) = 0$$

$$\vdots$$

$$\lim_{m\to\infty} d(g(F(x_m^n, x_m^1, ..., x_m^{n-1})), F(gx_m^n, gx_m^1, ..., gx_m^{n-1})) = 0.$$

As $F$ is continuous, therefore

$$F(gx_m^1, gx_m^2, ..., gx_m^n) \to F(x^1, x^2, ..., x^n)$$

$$F(gx_m^2, ..., gx_m^n, gx_m^1) \to F(x^2, ..., x^n, x^1)$$

$$\vdots$$

$$F(gx_m^n, gx_m^1, ..., gx_m^{n-1}) \to F(x^n, x^1, ..., x^{n-1}).$$

Using triangle inequality, we get

$$d(gx^1, F(gx_m^1, gx_m^2, ..., gx_m^n)) \leq d(gx^1, g(F(x_m^1, x_m^2, ..., x_m^n)))+$$
$$+d(g(F(x_m^1, x_m^2, ..., x_m^n)), F(gx_m^1, gx_m^2, ..., gx_m^n)).$$

By taking $m \to \infty$ and using continuity of $F$ and $g$, we have

$$d(gx^1, F(x^1, x^2, ..., x^n)) = 0, \text{ that is } gx^1 = F(x^1, x^2, ..., x^n),$$

and in a similar way, we have

$$gx^2 = F(x^2, ..., x^n, x^1), ..., gx^n = F(x^n, x^1, ..., x^{n-1}).$$

Thus $F$ and $g$ have an $n$-tupled coincidencve point.

If $g$ is an identity mapping in above theorem, we have the following result:

**Corollary 3.1.** *Let $(X, \preceq, d)$ be a complete partially ordered metric space and let $F : X^n \to X$ be a mapping. Suppose that the following hold:*
*(i) for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ if $x^1 \lesseqgtr F(x^1, x^2, ..., x^n)$, then*

$$F(x^1, x^2, ..., x^n) \lesseqgtr F(F(x^1, x^2, ..., x^n), y^2, y^3, ..., y^n);$$

*(ii) there exist $x_0^1, x_0^2, ..., x_0^n \in X$ such that $x_0^1 \lesseqgtr F(x_0^1, x_0^2, ..., x_0^n),\ x_0^2 \lesseqgtr F(x_0^2, ..., x_0^n, x_0^1),\ x_0^3 \lesseqgtr F(x_0^3, ..., x_0^n, x_0^1, x_0^2), ..., x_0^n \lesseqgtr F(x_0^n, x_0^1, ..., x_0^{n-1})$,*
*(iii) there exists $\alpha \in [0, 1)$ such that for all $x^1, x^2, ..., x^n,\ y^1, y^2, ..., y^n \in X$ for which $x^1 \lesseqgtr y^1,\ x^2 \lesseqgtr y^2, ..., x^n \lesseqgtr y^n$ with $x^1 \neq y^1, x^2 \neq y^2, ..., x^n \neq y^n$ satisfies,*

$$d(F(x^1, x^2, ..., x^n), F(y^1, y^2, ..., y^n)) \leq \alpha \max\{d(x^1, y^1), d(x^2, y^2), ..., d(x^n, y^n),$$
$$\frac{d(x^1, F(x^1, x^2, ..., x^n))d(y^1, F(y^1, y^2, ..., y^n))}{d(x^1, y^1)},$$
$$\frac{d(x^2, F(x^2, ..., x^n, x^1))d(y^2, F(y^2, ..., y^n, y^1))}{d(x^2, y^2)}, ...,$$

$$\frac{d(x^n, F(x^n, x^1, ..., x^{n-1}))d(y^n, F(y^n, y^1, ..., y^{n-1}))}{d(x^n, y^n)},$$
$$\frac{d(x^1, F(y^1, y^2, ..., y^n))d(y^1, F(x^1, x^2, ..., x^n))}{d(x^1, y^1)},$$
$$\frac{d(x^2, F(y^2, ..., y^n, y^1))d(y^2, F(x^2, ..., x^n, x^1))}{d(x^2, y^2)}, ...,$$
$$\frac{d(x^n, F(y^n, y^1, ..., y^{n-1}))d(y^n, F(x^n, x^1, ..., x^{n-1}))}{d(x^n, y^n)}\}, \tag{3.3}$$

*(iv) $F$ is continuous.*
*Then there exist $x^1, x^2, ..., x^n \in X$ such that $F(x^1, x^2, ..., x^n) = x^1, F(x^2, ..., x^n, x^1) = x^2, ..., F(x^n, x^1, ..., x^{n-1}) = x^n$, that is, $F$ has an $n$-tupled fixed point $(x^1, x^2, ..., x^n) \in X^n$.*

Now we shall prove the existence and uniqueness of $n$-tupled fixed point. Note that, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X^n$ with the following partial order relation: for $(x^1, x^2, ..., x^n), (y^1, y^2, ..., y^n) \in X^n$

$$(y^1, y^2, ..., y^n) \preceq (x^1, x^2, ..., x^n) \Leftrightarrow x^1 \preceq y^1, x^2 \succeq y^2, x^3 \preceq y^3, ..., x^n \succeq y^n.$$

**Theorem 3.2.** *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x^1, x^2, ..., x^n)$, $(z^1, z^2, ..., z^n) \in X^n$ there exists, $(y^1, y^2, ..., y^n) \in X^n$ such that $(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1}))$ is comparable to both $(F(x^1, x^2, ..., x^n), F(x^2, ..., x^n, x^1), ..., F(x^n, x^1, ..., x^{n-1}))$ and $F(z^1, z^2, ..., z^n), F(z^2, ..., z^n, z^1), ..., F(z^n, z^1, ..., z^{n-1}))$. Then $F$ and $g$ have a unique $n$-tupled common fixed point, that is, there exists $(u^1, u^2, ..., u^n) \in X^n$ such that*

$$u^1 = gu^1 = F(u^1, u^2, ..., u^n)$$
$$u^2 = gu^2 = F(u^2, ..., u^n, u^1)$$
$$\vdots$$
$$u^n = gu^n = F(u^n, u^1, ..., u^{n-1}).$$

**Proof.** From Theorem 3.1, the set of $n$-tupled coincidence points of $F$ and $g$ is non empty. Suppose that, $(x^1, x^2, ..., x^n)$ and $(z^1, z^2, ..., z^n)$ are two $n$-tupled coincidence points, that is,

$$F(x^1, x^2, ..., x^n) = gx^1, \quad F(z^1, z^2, ..., z^n) = gz^1$$
$$F(x^2, ..., x^n, x^1) = gx^2, \quad F(z^2, ..., z^n, z^1) = gz^2$$
$$\vdots$$
$$F(x^n, x^1, ..., x^{n-1}) = gx^n, \quad F(z^n, z^1, ..., z^{n-1}) = gz^n.$$

We shall show that

$$gx^1 = gz^1, gx^2 = gz^2, ..., gx^n = gz^n.$$

By assumption, there exists $(y^1, y^2, ..., y^n) \in X^n$ such that

$$(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1})),$$

is comparable to

$$(F(x^1, x^2, ..., x^n), F(x^2, ..., x^n, x^1), ..., F(x^n, x^1, ..., x^{n-1})),$$

and

$$(F(z^1, z^2, ..., z^n), F(z^2, ..., z^n, z^1), ..., F(z^n, z^1, ..., z^{n-1})).$$

Put $y_0^1 = y^1, y_0^2 = y^2, ..., y_0^n = y^n$ and choose $y_1^1, y_1^2, ..., y_1^n \in X$ such that

$$gy_1^1 = F(y_0^1, y_0^2, y_0^3, ..., y_0^n)$$
$$gy_1^2 = F(y_0^2, y_0^3, ..., y_0^n, y_0^1)$$
$$\vdots$$
$$gy_1^n = F(y_0^n, y_0^1, y_0^2, ..., y_0^{n-1}).$$

Then similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{gy_m^1\}, \{gy_m^2\}$, ..., $\{gy_m^n\}$ such that

$$gy_{m+1}^1 = F(y_m^1, y_m^2, y_m^3, ..., y_m^n)$$
$$gy_{m+1}^2 = F(y_m^2, y_m^3, ..., y_m^n, y_m^1)$$
$$\vdots$$
$$gy_{m+1}^n = F(y_m^n, y_m^1, y_m^2, ..., y_m^{n-1})\ \forall\ m \in \mathbb{N}.$$

Further set $x_0^1 = x^1, x_0^2 = x^2, ..., x_0^n = x^n$ and $z_0^1 = z^1, z_0^2 = z^2, ..., z_0^n = z^n$ and on the same way, define the sequences $\{gx_m^1\}, \{gx_m^2\}, ..., \{gx_m^n\}$ and $\{gz_m^1\}, \{gz_m^2\}, ..., \{gz_m^n\}$. Then as in Theorem 3.1, we can show that

$$gx_m^1 \to gx^1 = F(x^1, x^2, ..., x^n), \quad gz_m^1 \to gz^1 = F(z^1, z^2, ..., z^n)$$
$$gx_m^2 \to gx^2 = F(x^2, ..., x^n, x^1), \quad gz_m^2 \to gz^2 = F(z^2, ..., z^n, z^1)$$
$$\vdots$$
$$gx_m^n \to gx^n = F(x^n, x^1, ..., x^{n-1}), \quad gz_m^n \to gz^n = F(z^n, z^1, ..., z^{n-1})\ \forall\ m \in \mathbb{N}.$$

Since

$$\begin{aligned}&(F(x^1, x^2, x^3, ..., x^n), F(x^2, x^3, ..., x^n, x^1), ..., F(x^n, x^1, x^2, ..., x^{n-1}))\\ &\quad = (gx_1^1, gx_1^2, ..., gx_1^n) = (gx^1, gx^2, ..., gx^n),\end{aligned}$$

and

$$\begin{aligned}&(F(y^1, y^2, y^3, ..., y^n), F(y^2, y^3, ..., y^n, y^1), ..., F(y^n, y^1, y^2, ..., y^{n-1}))\\ &= (gy_1^1, gy_1^2, ..., gy_1^n),\end{aligned}$$

are comparable. Then we have

$$gx^1 \lesseqqgtr gy_1^1, gx^2 \lesseqqgtr gy_1^2, gx^3 \lesseqqgtr gy_1^3, ..., gx^n \lesseqqgtr gy_1^n,$$

and in a similar way, we have

$$gy_m^1 = F(y_{m-1}^1, y_{m-1}^2, ..., y_{m-1}^n) \lesseqqgtr F(x^1, x^2, ..., x^n) = gx^1$$
$$gy_m^2 = F(y_{m-1}^2, ..., y_{m-1}^n, y_{m-1}^1) \lesseqqgtr F(x^2, ..., x^n, x^1) = gx^2$$
$$\vdots$$
$$gy_m^n = F(y_{m-1}^n, y_{m-1}^1, ..., y_{m-1}^{n-1}) \lesseqqgtr F(x^n, x^1, ..., x^{n-1}) = gx^n.$$

Thus from (3.1) with $gx^1 \neq gy_m^1,\ gx^2 \neq gy_m^2, ..., gx^n \neq gy_m^n$, we have

$$\begin{aligned}d(gx^1, gy_{m+1}^1) &= d(F(x^1, x^2, ..., x^n), F(y_m^1, y_m^2, ..., y_m^n))\\ &\leq \alpha \max \Big\{ d(gx^1, gy_m^1), d(gx^2, gy_m^2), ..., d(gx^n, gy_m^n),\end{aligned}$$

$$\frac{d(gx^1, F(x^1, x^2, ..., x^n))d(gy_m^1, F(y_m^1, y_m^2.., ., y_m^n))}{d(gx^1, gy_m^1)},$$
$$\frac{d(gx^2, F(x^2, ..., x^n, x^1))d(gy_m^2, F(y_m^2.., ., y_m^n, y_m^1))}{d(gx^2, gy_m^2)},$$
$$\frac{d(gx^n, F(x^n, x^1, ..., x^{n-1}))d(gy_m^n, F(y_m^n, y_m^1.., ., y_m^{n-1}))}{d(gx^n, gy_m^n)},$$
$$\frac{d(gx^1, F(y_m^1, y_m^2.., ., y_m^n))d(gy_m^1, F(x^1, x^2, ..., x^n))}{d(gx^1, gy_m^1)},$$
$$\frac{d(gx^2, F(y_m^2.., ., y_m^n, y_m^1))d(gy_m^2, F(x^2, ..., x^n, x^1))}{d(gx^2, gy_m^2)},$$
$$\left.\frac{d(gx^n, F(y_m^n, y_m^1.., ., y_m^{n-1}))d(gy_m^n, F(x^n, x^1, ..., x^{n-1}))}{d(gx^n, gy_m^n)}\right\},$$
$$= \alpha \max\{d(gx^1, gy_m^1), d(gx^2, gy_m^2), ..., d(gx^n, gy_m^n), d(gx^1, gy_{m+1}^1), d(gx^2, gy_{m+1}^2), ..., d(gx^n, gy_{m+1}^n)\}. \quad (3.4)$$

Similarly, we can prove that

$$d(gx^2, gy_{m+1}^2) \le \alpha \max\{d(gx^2, gy_m^2), ..., d(gx^n, gy_m^n), d(gx^1, gy_m^1), d(gx^2, gy_{m+1}^2), ..., d(gx^n, gy_{m+1}^n), d(gx^1, gy_{m+1}^1)\}$$
$$\vdots$$
$$d(gx^n, gy_{m+1}^n) \le \alpha \max\{d(gx^n, gy_m^n), d(gx^1, gy_m^1), ..., d(gx^{n-1}, gy_m^{n-1}), d(gx^n, gy_{m+1}^n), d(gx^1, gy_{m+1}^1), ..., d(gx^{n-1}, gy_{m+1}^{n-1})\}.$$

Hence

$$\max\{d(gx^1, gy_{m+1}^1), d(gx^2, gy_{m+1}^2), ..., d(gx^n, gy_{m+1}^n)\} \le \alpha \max\{d(gx^1, gy_m^1), d(gx^2, gy_m^2), ..., d(gx^n, gy_m^n)\},$$

and by induction,

$$\max\{d(gx^1, gy_{m+1}^1), d(gx^2, gy_{m+1}^2), ..., d(gx^n, gy_{m+1}^n)\} \le \alpha^m \max\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n)\}.$$

On taking limit $m \to \infty$, we get

$$\lim_{m\to\infty} d(gx^1, gy_{m+1}^1) = 0, \ \lim_{m\to\infty} d(gx^2, gy_{m+1}^2) = 0, ..., \lim_{m\to\infty} d(gx^n, gy_{m+1}^n) = 0.$$

Similarly we can prove that

$$\lim_{m\to\infty} d(gz^1, gy_{m+1}^1) = 0, \ \lim_{m\to\infty} d(gz^2, gy_{m+1}^2) = 0, ..., \lim_{m\to\infty} d(gz^n, gy_{m+1}^n) = 0.$$

Finally, we have

$$d(gx^1, gz^1) \le d(gx^1, gx_m^1) + d(gx_m^1, gz^1)$$
$$d(gx^2, gz^2) \le d(gx^2, gx_m^2) + d(gx_m^2, gz^2)$$
$$\vdots$$
$$d(gx^n, gz^n) \le d(gx^n, gx_m^n) + d(gx_m^n, gz^n).$$

Taking $m \to \infty$ in these inequalities, we get $d(gx^1, gz^1) = d(gx^2, gz^2) = ... = d(gx^n, gz^n) = 0$, that is,

$$gx^1 = gz^1, \ gx^2 = gz^2, ..., gx^n = gz^n.$$

Denote $gx^1 = p^1,\ gx^2 = p^2, ..., gx^n = p^n$, we have that

$$gp^1 = g(gx^1) = g(F(x^1, x^2, ..., x^n))$$
$$gp^2 = g(gx^2) = g(F(x^2, ..., x^n, x^1))$$
$$\vdots$$
$$gp^n = g(gx^n) = g(F(x^n, x^1, ..., x^{n-1})).$$

By the definition of sequences $\{x_m^1\}, \{x_m^2\}, ..., \{x_m^n\}$, we have

$$gx_m^1 = F(x_{m-1}^1, x_{m-1}^2, ..., x_{m-1}^n)$$
$$gx_m^2 = F(x_{m-1}^2, ..., x_{m-1}^n, x_{m-1}^1)$$
$$\vdots$$
$$gx_m^n = F(x_{m-1}^n, x_{m-1}^1, ..., x_{m-1}^{n-1}),$$

and so

$$gx_m^1 \to F(x^1, x^2, ..., x^n),\ gx_m^2 \to F(x^2, ..., x^n, x^1), ..., gx_m^n \to F(x^n, x^1, ..., x^{n-1}).$$

Compatibility of $F$ and $g$ implies that

$$\lim_{m\to\infty} d(g(F(x_m^1, x_m^2, ..., x_m^n)), F(gx_m^1, gx_m^2, ..., gx_m^n)) \to 0,$$

that is,

$$g(F(x^1, x^2, ..., x^n)) = F(gx^1, gx^2, ..., gx^n).$$

Then $gp^1 = F(p^1, p^2, ..., p^n)$ and similarly,
$gp^2 = F(p^2, ..., p^n, p^1), ..., gp^n = F(p^n, p^1, ..., p^{n-1})$. Thus $(p^1, p^2, ..., p^n)$ is an $n$-tupled coincidence point. Thus, it follows $gp^1 = gx^1,\ gp^2 = gx^2, ..., gp^n = gx^n$, that is, $gp^1 = p^1,\ gp^2 = p^2, ..., gp^n = p^n$. Hence

$$p^1 = gp^1 = F(p^1, p^2, ..., p^n),$$
$$p^2 = gp^2 = F(p^2, ..., p^n, p^1),$$
$$\vdots$$
$$p^n = gp^n = F(p^n, p^1, ..., p^{n-1}).$$

Therefore $(p^1, p^2, ..., p^n)$ is an $n$-tupled common fixed point of $F$ and $g$. To prove the uniqueness, assume that $(q^1, q^2, ..., q^n)$ is another $n$-tupled common fixed point. Then as above we have

$$q^1 = gq^1 = gp^1 = p^1,$$
$$q^2 = gq^2 = gp^2 = p^2,$$
$$\vdots$$
$$q^n = gq^n = gp^n = p^n.$$

Hence, we get the result.

**Example 3.1.** Let $X = [0, 1]$. Then $(X, d, \preceq)$ is a partially ordered set with the natural ordering $\preceq$ of real numbers and natural metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $g : X \to X$ by $g(x) = x^2$ for all $x \in X$ and $F : X^n \to X$ (wherein $n$ is fixed and $n > 1$) by

$$F(x^1, x^2, ..., x^n) = \frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + ..... + (x^{n-1})^2 + (x^n)^2}{2n},$$

for all $x^1, x^2, ..., x^n \in X$. All the conditions of Theorem 3.1 are satisfied, the contractive condition (for $\alpha = \frac{1}{2}$), follows from

$$
\begin{aligned}
&d(F(x^1, x^2, x^3, ..., x^n), F(y^1, y^2, y^3, ..., y^n)) \\
&= d\Big(\frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + .... + (x^n)^2}{2n}, \frac{(y^1)^2 + (y^2)^2 + (y^3)^2 + .... + (y^n)^2}{2n}\Big) \\
&= \left|\frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + .... + (x^n)^2}{2n} - \frac{(y^1)^2 + (y^2)^2 + (y^3)^2 + .... + (y^n)^2}{2n}\right| \\
&= \left|\frac{((x^1)^2 - (y^1)^2) + ((x^2)^2 - (y^2)^2) + ((x^3)^2 - (y^3)^2) + .... + ((x^n)^2 - (y^n)^2)}{2n}\right| \\
&\le \frac{|(x^1)^2 - (y^1)^2| + |(x^2)^2 - (y^2)^2| + |(x^3)^2 - (y^3)^2| + .... + |(x^n)^2 - (y^n)^2|}{2n} \\
&\le \frac{1}{2n}\Big[n \max\Big\{|(x^1)^2 - (y^1)^2|, |(x^2)^2 - (y^2)^2|, |(x^3)^2 - (y^3)^2|, ...., |(x^n)^2 - (y^n)^2|\Big\}\Big] \\
&= \frac{1}{2}\Big[\max\Big\{|gx^1 - gy^1|, |gx^2 - gy^2|, |gx^3 - gy^3|, ...., |gx^n - gy^n|\Big\}\Big] \\
&= \frac{1}{2}\Big[\max\Big\{d(gx^1, gy^1), d(gx^2, gy^2), d(gx^3, gy^3), ...., d(gx^n, gy^n)\Big\}\Big] \\
&\le \alpha \max\Big\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^n, gy^n), \frac{d(gx^1, F(x^1, x^2, ..., x^n))d(gy^1, F(y^1, y^2, ..., y^n))}{d(gx^1, gy^1)}, \\
&\frac{d(gx^2, F(x^2, ..., x^n, x^1))d(gy^2, F(y^2, ..., y^n, y^1))}{d(gx^2, gy^2)}, ..., \frac{d(gx^n, F(x^n, x^1, ..., x^{n-1}))d(gy^n, F(y^n, y^1, ..., y^{n-1}))}{d(gx^n, gy^n)}, \\
&\frac{d(gx^1, F(y^1, y^2, ..., y^n))d(gy^1, F(x^1, x^2, ..., x^n))}{d(gx^1, gy^1)}, \frac{d(gx^2, F(y^2, ..., y^n, y^1))d(gy^2, F(x^2, ..., x^n, x^1))}{d(gx^2, gy^2)}, ..., \\
&\frac{d(gx^n, F(y^n, y^1, ..., y^{n-1}))d(gy^n, F(x^n, x^1, ..., x^{n-1}))}{d(gx^n, gy^n)}\Big\}.
\end{aligned}
$$

Hence, all the conditions of Theorem 3.1 are satisfied and $(0, 0, ..., 0)$ is an $n$-tupled coincidence point of $F$ and $g$.

# 4 Conclusions

In this paper, we present some $n$-tupled coincidence point results (for even $n$) for a pair of mappings without mixed monotone property satisfying a contractive condition of rational type in metric spaces equipped with a partial ordering. Also some results on the existence and uniqueness of $n$-tupled common fixed points are proved.

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The authors declare that no competing interests exist.

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