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Research Article

M-Lump Solution, Semirational Solution, and Self-Consistent Source Extension of a Novel $(2+1)$-Dimensional KdV Equation

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In this paper, we derive the M-lump solution in terms of Matsuno determinant for the combined KP3 and KP4 (cKP3-4) equation by applying the double-sum identities for determinant and investigate the dynamical behaviors of 1- and 2-lump solutions. In addition, we derive the Grammian solution for the cKP3-4 equation and construct the semirational solutions from the Grammian solution. Through the asymptotic analysis, we show that the semirational solutions describe fusion and fission of lumps and line solitons and rogue lump phenomena. Furthermore, we construct the cKP3-4 equation with self-consistent sources via the source generation procedure and present its Grammian and Wronskian solution.

1. Introduction

The Korteweg-de Vries (KdV) equation plays an important role in the development of the soliton theory. In 1895, Korteweg and de Vries derived the KdV equation to model moderately small shallow-water waves [1]. They also presented a large set of permanent wave solutions including solitary wave solution for the KdV equation. In 1965, Zabusky and Kruskal discovered the remarkable particle-like behavior of solitary wave solutions to the KdV equation [2]. In 1967, Gardner et al. invented the inverse scattering transform to solve the Cauchy problem for the KdV equation which leads to the discovery of the integrable systems soon afterwards [3]. Besides modeling shallow-water waves, the KdV equation has arisen in the study of stratified internal waves, nonlinear acoustic waves, plasma physics, lattice dynamics, geophysics, quantum field theory, string and conformal field theory, etc. [4–9]. The KdV equation in nondimensional form is

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1}$$

Several $(2+1)$-dimensional generalizations of the KdV equation including Kadomtsev-Petviashvili (KP) equation [10], Date-Jimbo-Kashiwara-Miwa (DJKM) equation [11], Nizhnik-Novikov-Veselov (NNV) equation [12–14], Boiti-Leon-Manna-Pempinelli equation [15], Ito equation [16], and Bogoyavlenskii's breaking soliton equation [17] have been derived. In Ref. [18], the authors proposed a novel integrable $(2+1)$-dimensional extension of the KdV equation which is a combination of the KP equation and the DJKM equation, called the combined KP3 and KP4 (cKP3-4) equation which is written as

$$u_t = a\left(6uu_x + u_{xxx} - 3W_y\right) + b\left(2wu_x - z_y + u_{xxy} + 4uu_y\right), u_y = w_x, u_{yy} = z_{xx}. \tag{2}$$

The cKP3-4 equation is physically interesting because it exhibits line soliton molecules involving any number of line solitons, but the KP equation and DJKM equation do not have line soliton molecules. Furthermore, the cKP3-4 equation possesses the D'Alembert-type solutions including various new types of solitons and soliton molecules.

Lump wave is a kind of multidimensional localized wave decaying algebraically in all directions in the space. In 1977, Manakov et al. derived the analytical lump solutions of the

KP1 equation applying inverse scattering method [19]. In [20], the authors developed a method to obtain lump solutions to the soliton equations by taking the long wave limits of the N-soliton solutions. Since then, lump solutions for the numerous nonlinear evolution equations are constructed through inverse scattering method, Darboux transformation, Bäcklund transformation, long wave limit method, Hirota bilinear method and symbolic computation, etc. [21–25]. In this paper, we apply the Hirota bilinear method and determinant technique to derive the M-lump solution in terms of Matsuno determinant for the cKP3-4 equation.

Semirational solution describing resonant collision between multiple waves can be obtained from Grammian solution. The solitary waves and their resonant interaction can be used to study interesting phenomena in the realistic model such as web-shaped waveforms. As one case of resonant interactions, the resonant collision between lumps and line solitons is first studied in [26] and has attracted intensive attention [27–29]. The resonant collision of lumps and line solitons describes the phenomena of merging of lumps and line solitons into line solitons or detaching of lumps and line solitons from line solitons. Another interesting phenomenon appears for the resonant collision of lumps and line solitons when lumps are emitted from a line soliton and then merge with remaining solitons after a period of time, which is called the "rogue lump" [30–32]. In this paper, we construct the semirational solution for the cKP3-4 equation from its Grammian solution and discuss the resonant collision between lumps and solitons.

The soliton equations with self-consistent sources model various physically interesting processes. These kinds of systems are usually applied to describe interactions between different solitary waves and have important applications in hydrodynamics, plasma physics, and nonlinear optics [33–43]. For example, the KP equation with a self-consistent source [38–40]

$$\begin{gathered} u_t + 6uu_x + u_{xxx} - 3\partial_x^{-1}u_{yy} = -\kappa|\phi|_x^2, \\ si\phi_y = \phi_{xx} + u\phi, \end{gathered} \tag{3}$$

models the interaction of a long wave with a short-wave packet propagating on the x, y plane at an angle to each other, where $u(x, y, t)$ is the long wave amplitude, $\phi(x, y, t)$ is the complex short-wave envelope, and the parameter κ satisfies $\kappa^2 = 1$. The solutions for the soliton equations with self-consistent sources have been derived by applying various methods such as inverse scattering methods [38–40], Darboux transformation methods [44–46], Hirota's bilinear method and Wronskian technique [47–50], and deformations of binary Darboux transformations [51, 52]. A new algebraic method, called the source generation procedure, has been proposed in Ref. [53] to construct and solve the soliton equations with self-consistent sources in a systematic way. In this paper, we construct the cKP3-4 equation with self-consistent sources by applying the source generation procedure and derive its Wronskian and Grammian solution.

The structure of this paper is as follows. In Section 2, we present the M-lump solution in the form of Matsuno determinant for the cKP3-4 equation and investigate the dynamics of 1- and 2- lump solution. In Section 3, we first derive the Grammian solution for the cKP3-4 equation and then construct the semirational solution from the Grammian solution. We also illustrate the resonant collision between lumps and solitons. In Section 4, we construct the cKP3-4 equation with self-consistent sources by applying the source generation procedure and derive its Grammian and Wronskian solution. A conclusion and discussion are given in Section 5.

2. M-Lump Solution for the cKP3-4 Equation

In this section, we construct the M-lump solution expressed in the form of Matsuno determinant for the cKP3-4 equation and prove the M-lump solution satisfies the bilinear cKP3-4 equations (4) and (5) by utilizing the double-sum identities for determinant [54]. We also analyze the dynamics of the 1- and 2-lump solutions for the cKP3-4 equation.

Through the dependent variable transformations $u = 2 \ln (F)_{xx}$, $w = 2 \ln (F)_{xy}$, $z = 2 \ln (F)_{yy}$ and introducing auxiliary variable τ, the cKP3-4 equation (2) can be transformed into the bilinear form [18]

$$\left[D_x D_\tau + a\left(3D_y^2 - D_x^4\right)\right]\Big) F \cdot F = 0, \tag{4}$$

$$\left[a\left(2bD_x^3 D_y - 3D_x D_t + 3D_x D_\tau\right) + bDyD_\tau\right] F \cdot F = 0. \tag{5}$$

Proposition 1. *Nth-order rational solutions of cKP3-4 equations (4) and (5) can be expressed in the following determinant form:*

$$F = \begin{vmatrix} \theta_1 & \dfrac{2i}{p_1 - p_2} & \cdots & \dfrac{2i}{p_1 - p_N} \\ \dfrac{2i}{p_2 - p_1} & \theta_2 & \cdots & \dfrac{2i}{p_2 - p_N} \\ \vdots & \vdots & & \vdots \\ \dfrac{2i}{p_N - p_1} & \dfrac{2i}{p_N - p_2} & \cdots & \theta_N \end{vmatrix}, \tag{6}$$

where $\theta_j = x + p_j y - (3ap_j^2 + bp_j^3)t - 3ap_j^2\tau + \theta_j^{(0)} (j = 1 \cdots N)$ *in which* $p_j (j = 1 \cdots N)$ *are complex parameters and* $\theta_j^{(0)} (j = 1 \cdots N)$ *are arbitrary complex constants. Furthermore, if we take* $N = 2M$ *(M is a positive integer) and* $p_j^* = p_{M+j}$, $(\theta_j^{(0)})^* = \theta_{M+j}^{(0)} (j = 1 \cdots M)$ *in (6), we obtain the M-lump solution of the cKP3-4 equation.*

The proof of Proposition 1 is given in Appendix A.

The M-lump solution given in Proposition 1 can be written in the following form:

$$F_M = \begin{vmatrix} C & A \\ -A^* & (C^*)^T \end{vmatrix}, \quad (7)$$

where $()^T$ denotes the transpose of the matrix; C and A are $M \times M$ matrices defined by

$$A = \begin{pmatrix} \frac{2i}{p_1 - p_1^*} & \frac{2i}{p_1 - p_2^*} & \cdots & \frac{2i}{p_1 - p_M^*} \\ \frac{2i}{p_2 - p_1^*} & \frac{2i}{p_2 - p_2^*} & \cdots & \frac{2i}{p_2 - p_M^*} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{2i}{p_M - p_1^*} & \frac{2i}{p_M - p_2^*} & \cdots & \frac{2i}{p_M - p_M^*} \end{pmatrix}, \quad (8)$$

$$C = \begin{pmatrix} \theta_1 & \frac{2i}{p_1 - p_2} & \cdots & \frac{2i}{p_1 - p_M} \\ \frac{2i}{p_2 - p_1} & \theta_2 & \cdots & \frac{2i}{p_2 - p_M} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{2i}{p_M - p_1} & \frac{2i}{p_M - p_2} & \cdots & \theta_M \end{pmatrix}, \quad (9)$$

in which $\theta_j = x + p_j y - (3ap_j^2 + bp_j^3)t - 3ap_j^2\tau + \theta_j^{(0)}$ for $j = 1, 2, \cdots, M$. It is known that the determinant given in (7) is positive [19, 20]. Therefore, M-lump solution $u = 2(\ln F_M)_{xx}$ is the nonsingular rational solution.

If we take $M = 1$ in (7), we obtain the following 1-lump solution for cKP3-4 equations (4) and (5):

$$F_1 = \begin{vmatrix} \theta_1 & \frac{2i}{p_1 - p_1^*} \\ \frac{2i}{p_1^* - p_1} & \theta_1^* \end{vmatrix} = |\theta_1|^2 - \frac{4}{(p_1 - p_1^*)^2}, \theta_1 = x + p_1 y - (3ap_1^2 + bp_1^3)t - 3ap_1^2\tau + \theta_1^{(0)}, \quad (10)$$

which is real and positive. By calculating the local maximum value of the multivariable function $u = 2(\ln F_1)_{xx}$, we obtain the trajectory $[x(t), y(t)]$ for the peak of 1-lump:

$$\theta_{1,R} = 0, \theta_{1,I} = 0, \quad (11)$$

where $\theta_{1,R}$ and $\theta_{1,I}$ denote the real and imaginary part of θ_1, respectively, and

$$\begin{aligned} \theta_{1,R} &= x + \alpha_1 y - \left[3a(\alpha_1^2 - \beta_1^2) + b\alpha_1(\alpha_1^2 - 3\beta_1^2)\right]t - 3a(\alpha_1^2 - \beta_1^2)\tau + \theta_{1,R}^{(0)}, \\ \theta_{1,I} &= \beta_1 y - \left[6a\alpha_1\beta_1 + b\beta_1(3\alpha_1^2 - \beta_1^2)\right]t - 6a\alpha_1\beta_1\tau + \theta_{1,I}^{(0)}, \end{aligned} \quad (12)$$

in which $\alpha_1 = \mathrm{Re}\,(p_1), \beta_1 = \mathrm{Im}\,(p_1), \theta_{1,R}^{(0)} = \mathrm{Re}\,(\theta_1), \theta_{1,I}^{(0)} = \mathrm{Im}\,(\theta_1)$. Figure 1 shows the 1-lump solution $u = 2\ln(F)_{xx}$ on the (x, y) plane by taking $t = 0.5, \tau = 0, a = 1, b = 1, p_1 = 1 + 0.9i, \theta_1^{(0)} = 0$.

If we take $M = 2$ in (7), we obtain the following 2-lump solution for cKP3-4 equations (4) and (5):

$$F_2 = \begin{vmatrix} \theta_1 & \frac{2i}{p_1 - p_2} & \frac{2i}{p_1 - p_1^*} & \frac{2i}{p_1 - p_2^*} \\ \frac{2i}{p_2 - p_1} & \theta_2 & \frac{2i}{p_2 - p_1^*} & \frac{2i}{p_2 - p_2^*} \\ \frac{2i}{p_1^* - p_1} & \frac{2i}{p_1^* - p_2} & \theta_1^* & \frac{2i}{p_1^* - p_2^*} \\ \frac{2i}{p_2^* - p_1} & \frac{2i}{p_2^* - p_2} & \frac{2i}{p_2^* - p_1^*} & \theta_2^* \end{vmatrix}, \quad (13)$$

where $\theta_j = x + p_j y - (3ap_j^2 + bp_j^3)t - 3ap_j^2\tau + \theta_j^{(0)}\,(j = 1, 2)$.

By expanding the determinant in (13), we obtain

$$\begin{aligned} F_2 = {} & |\theta_1|^2|\theta_2|^2 - \frac{4}{(p_2 - p_2^*)^2}|\theta_1|^2 - \frac{4}{(p_1 - p_1^*)^2}|\theta_2|^2 \\ & - \frac{4}{(p_1^* - p_2^*)^2}\theta_1\theta_2 - \frac{4}{(p_1 - p_2)^2}\theta_1^*\theta_2^* - \frac{4}{(p_1^* - p_2)^2}\theta_1\theta_2^* \\ & - \frac{4}{(p_1 - p_2^*)^2}\theta_1^*\theta_2 + \frac{16}{(p_1 - p_2)^2(p_1^* - p_2^*)^2} \\ & + \frac{16}{(p_1 - p_1^*)^2(p_2 - p_2^*)^2} + \frac{16}{(p_1 - p_2^*)^2(p_2 - p_1^*)^2}. \end{aligned} \quad (14)$$

For the asymptotic analysis of lump 1, when $\theta_{1,R} \approx 0$, $\theta_{1,I} \approx 0$, because $|\theta_2| = O(t)$ as $t \longrightarrow \pm\infty$, we obtain the asymptotic form of lump 1 which is denoted as $F_2^{(1)}$ from equation (14):

$$F_2^{(1)} \sim \left(|\theta_1|^2 - \frac{4}{(p_1 - p_1^*)^2}\right)|\theta_2|^2 \text{ as } t \longrightarrow \pm\infty, \quad (15)$$

which is equivalent to

$$F_2^{(1)} \sim |\theta_1|^2 - \frac{4}{(p_1 - p_1^*)^2} \text{ as } t \longrightarrow \pm\infty, \quad (16)$$

by noticing $u = 2\ln(F)_{xx}$. In the same way, we can derive the asymptotic form of lump 2 which is denoted as $F_2^{(2)}$ from equation (14):

$$F_2^{(2)} \sim |\theta_2|^2 - \frac{4}{(p_2 - p_2^*)^2} \; as \; t \longrightarrow \pm\infty. \quad (17)$$

We conclude from above asymptotic analysis that 2-lump solution (14) describes the elastic interaction between two lumps and the phase shifts of two lumps during the interaction are zero. Figure 2 shows the elastic interaction between two lumps on the (x, y) plane at different times by

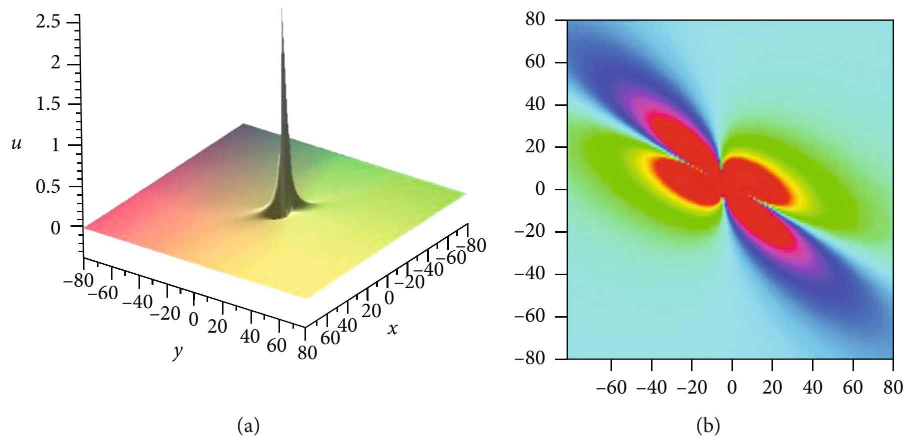


FIGURE 1: (a) The 1-lump solution u for cKP3-4 equation (2). (b) Density plot of (a).

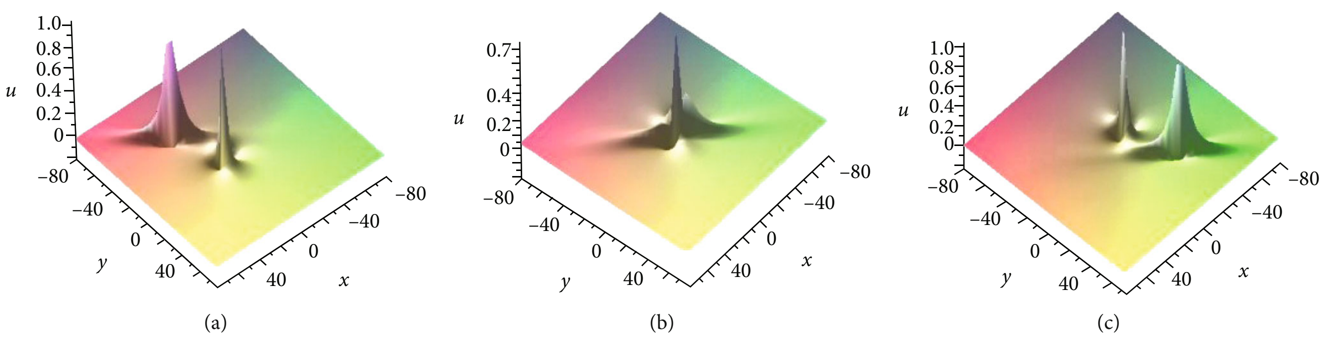


FIGURE 2: The interaction of two lumps for cKP3-4 equation (2): (a) $t = -4$, (b) $t = 0.5$, and (c) $t = 4$.

taking $\tau = 0, a = 1, b = 1, p_1 = -0.1 + 0.9i, p_2 = 1 - 0.5i, \theta_1^{(0)} = \theta_2^{(0)} = 0$.

3. Grammian and Semirational Solution of the cKP3-4 Equation

In this section, we derive the Grammian solution for the cKP3-4 equation and construct the semirational solution from the Grammian solution. We also illustrate the several semirational solutions graphically.

Proposition 2. *cKP3-4 equations (4) and (5) possess the following Grammian solution:*

$$G = \det |a_{rs}|_{1\le r,s\le N}, a_{rs} = C_{rs} + \int f_r g_s d_x, \quad (18)$$

where $f_r, g_s (r, s = 1,\cdots,N)$ are functions of x, y, t, τ and f_r, g_s $(r, s = 1,\cdots,N)$ satisfy the following dispersion relations:

$$\frac{\partial f_r}{\partial y} = -i\frac{\partial^2 f_r}{\partial x^2}, \frac{\partial f_r}{\partial \tau} = 4a\frac{\partial^3 f_r}{\partial x^3}, \frac{\partial f_r}{\partial t} = 4a\frac{\partial^3 f_r}{\partial x^3} - 2bi\frac{\partial^4 f_r}{\partial x^4}, \quad (19)$$

$$\frac{\partial g_s}{\partial y} = i\frac{\partial^2 g_s}{\partial x^2}, \frac{\partial g_s}{\partial \tau} = 4a\frac{\partial^3 g_s}{\partial x^3}, \frac{\partial g_s}{\partial t} = 4a\frac{\partial^3 g_s}{\partial x^3} + 2bi\frac{\partial^4 g_s}{\partial x^4}. \quad (20)$$

The proof of Proposition 2 is given in Appendix B.

To obtain the N-soliton solution for the cKP3-4 equation, we take

$$f_r = e^{\xi_r}, \xi_r = p_r x - ip_r^2 y + \left(4ap_r^3 - 2bip_r^4\right)t + 4ap_r^3\tau + \xi_{r0}(r = 1,\cdots,N), \quad (21)$$

$$g_s = e^{\eta_s}, \eta_s = q_s x + iq_s^2 y + \left(4aq_s^3 + 2biq_s^4\right)t + 4aq_s^3\tau + \eta_{s0}(s = 1,\cdots,N), \quad (22)$$

in the Grammian solution (18), where $p_r, q_s, \xi_{r0}, \eta_{s0}$ are arbitrary complex constants. Furthermore, to construct the semirational solution for the cKP3-4 equation, we introduce differential operators A_r, B_s as [55]

$$A_r = \sum_{k=0}^{n_r} c_{rk}\left(p_r\partial_{p_r}\right)^{n_r-k}, B_s = \sum_{l=0}^{n_s} d_{sl}\left(q_s\partial_{q_s}\right)^{n_s-l}, \quad (23)$$

where c_{rk}, d_{sl} are constants and n_r, n_s are positive integers. If we choose f_r, g_s as

$$f_r = A_r e^{\xi_r}, g_s = B_s e^{\eta_s} (r, s = 1 \cdots N), \tag{24}$$

in (18), where e^{ξ_r} and e^{η_r} $(r, s = 1, \cdots, N)$ are given in (21) and (22), we obtain the following semirational solution for the cKP3-4 equation

$$G = \det \left| C_{rs} + e^{(\xi_r + \eta_s)} \sum_{k=0}^{n_r} c_{rk} \left(\xi'_r + p_r \partial_{p_r} \right)^{n_r - k} \times \sum_{l=0}^{n_s} d_{sl} \left(\eta'_s + q_s \partial_{q_s} \right)^{n_s - l} \frac{1}{p_r + q_s} \right|_{1 \le r,s \le N}, \tag{25}$$

where $\xi'_r = p_r x - 2ip_r^2 y + (12ap_r^3 - 8bip_r^4)t + 12ap_r^3 \tau, \eta'_s = q_s x + 2iq_s^2 y + (12aq_s^3 + 8biq_s^4)t + 12aq_s^3 \tau (r, s = 1, \cdots, N)$. The fundamental semirational solution which is obtained by taking $N = 1$, $n_1 = 1$ in (25) is written as

$$G = C_{11} + e^{(\xi_1 + \eta_1)} \left[\left(\xi'_1 + c_{11} - \frac{p_1}{p_1 + q_1} \right) \left(\eta'_1 + d_{11} - \frac{q_1}{p_1 + q_1} \right) + \frac{p_1 q_1}{(p_1 + q_1)^2} \right] \frac{1}{p_1 + q_1}, \tag{26}$$

where we have taken $c_{10} = d_{10} = 1$. Furthermore, If we take $p_1 = q_1^*$, $c_{11} = d_{11}^*$ in (26), then through the dependent variable transformation $u = 2 \ln (G)_{xx}$, we obtain

$$u = 2 \frac{-2l_1^2 + 2l_2^2 + \left(1/2p_{1,R}^2\right) + 2|p_1|^2 C_{11} e^{-\Gamma} \left(p_{1,R} l_1^2 + p_{1,R} l_2^2 + 2l_1 + \left(3/4p_{1,R}\right) \right)}{\left[\left(l_1^2 + l_2^2 + \left(1/4p_{1,R}^2\right) \right) \left(|p_1|^2 / 2p_{1,R} \right) + C_{11} e^{-\Gamma} \right]^2} (C_{11} \neq 0), \tag{27}$$

where

$$\begin{aligned} l_1 &= x + 2p_{1,I} y + \left[12a\left(p_{1,R}^2 - p_{1,I}^2\right) + 8b\left(3p_{1,R}^2 p_{1,I} - p_{1,I}^3\right) \right] t \\ &\quad + 12a\left(p_{1,R}^2 - p_{1,I}^2\right)\tau + \text{Re}\left(\frac{c_{11}}{p_1}\right) - \frac{1}{2p_{1,R}}, \\ l_2 &= -2p_{1,R} y + \left[24ap_{1,R} p_{1,I} - 8b\left(p_{1,R}^3 - 3p_{1,R} p_{1,I}^2\right) \right] t \\ &\quad + 24ap_{1,R} p_{1,I} \tau + \text{Im}\left(\frac{c_{11}}{p_1}\right), \\ \Gamma &= 2p_{1,R} x + 4p_{1,R} p_{1,I} y + \left[8a\left(p_{1,R}^3 - 3p_{1,R} p_{1,I}^2\right) \right. \\ &\quad \left. + 4b\left(4p_{1,R}^3 p_{1,I} - 4p_{1,R} p_{1,I}^3\right) \right] t + 8a\left(p_{1,R}^3 - 3p_{1,R} p_{1,I}^2\right)\tau \\ &\quad + 2\,\text{Re}\,(\xi_{10}). \end{aligned} \tag{28}$$

The fundamental semirational solution (27) describes the resonant collision between one lump and one soliton, in which the peak of lump moves along the trajectory $[x(t), y(t)]$:

$$l_1 = 0, l_2 = 0. \tag{29}$$

And the lump reaches maximum amplitude $16p_{1,R}^2$ at point A_1 and attains minimum amplitude $-2p_{1,R}^2$ at the points A_2, A_3, in which

$$\begin{aligned} &A_1 \left(\left[-12a\left(p_{1,R}^2 + p_{1,I}^2\right) - 8bp_{1,I}\left(2p_{1,R}^2 + 2p_{1,I}^2\right) \right] t - 12a\left(p_{1,R}^2 + p_{1,I}^2\right)\tau - \frac{p_{1,I}}{p_{1,R}} \text{Im}\left(\frac{c_{11}}{p_1}\right) \right. \\ &\quad \left. - \text{Re}\left(\frac{c_{11}}{p_1}\right) + \frac{1}{2p_{1,R}}, \left[12ap_{1,I} - 4b\left(p_{1,R}^2 - 3p_{1,I}^2\right) \right] t + 12ap_{1,I}\tau + \frac{1}{2p_{1,R}} \text{Im}\left(\frac{c_{11}}{p_1}\right) \right), \\ &A_2 \left(\left[-12a\left(p_{1,R}^2 + p_{1,I}^2\right) - 8bp_{1,I}\left(2p_{1,R}^2 + 2p_{1,I}^2\right) \right] t - 12a\left(p_{1,R}^2 + p_{1,I}^2\right)\tau - \frac{p_{1,I}}{p_{1,R}} \text{Im}\left(\frac{c_{11}}{p_1}\right) \right. \\ &\quad \left. - \text{Re}\left(\frac{c_{11}}{p_1}\right) + \frac{1}{2p_{1,R}} + \frac{\sqrt{3}}{2p_{1,R}}, \left[12ap_{1,I} - 4b\left(p_{1,R}^2 - 3p_{1,I}^2\right) \right] t + 12ap_{1,I}\tau + \frac{1}{2p_{1,R}} \text{Im}\left(\frac{c_{11}}{p_1}\right) \right), \\ &A_3 \left(\left[-12a\left(p_{1,R}^2 + p_{1,I}^2\right) - 8bp_{1,I}\left(2p_{1,R}^2 + 2p_{1,I}^2\right) \right] t - 12a\left(p_{1,R}^2 + p_{1,I}^2\right)\tau - \frac{p_{1,I}}{p_{1,R}} \text{Im}\left(\frac{c_{11}}{p_1}\right) \right. \\ &\quad \left. - \text{Re}\left(\frac{c_{11}}{p_1}\right) + \frac{1}{2p_{1,R}} - \frac{\sqrt{3}}{2p_{1,R}}, \left[12ap_{1,I} - 4b\left(p_{1,R}^2 - 3p_{1,I}^2\right) \right] t + 12ap_{1,I}\tau + \frac{1}{2p_{1,R}} \text{Im}\left(\frac{c_{11}}{p_1}\right) \right). \end{aligned} \tag{30}$$

The maximum amplitude of the lump in the fundamental semirational solution (27) is

$$\begin{aligned} u|_{A_1} &= u_{\max} = 2 \frac{\left(1/2p_{1,R}^2\right) + 2C_{11}|p_1|^2 \left(3/4p_{1,R}\right) e^{-\Gamma_{\max}}}{\left[C_{11} e^{-\Gamma_{\max}} + \left(|p_1|^2 / 8p_{1,R}^3 \right) \right]^2}, \\ \Gamma_{\max} &= -p_{1,R}^3 16\left(a + 2bp_{1,I}\right)t - 16ap_{1,R}^3 \tau + \text{Re}\,(\xi_{10}) - 2p_{1,R}\,\text{Re}\left(\frac{c_{11}}{p_1}\right) + 1, \end{aligned} \tag{31}$$

and the minimum amplitudes of the lump in the fundamental semirational solution (27) are

$$\begin{aligned} u|_{A_2} &= u_{\min 1} = 2 \frac{-\left(1/p_{1,R}^2\right) + 2C_{11}|p_1|^2 \left(3 + 2\sqrt{3}/2p_{1,R}\right) e^{-\Gamma_{\min 1}}}{\left[C_{11} e^{-\Gamma_{\min 1}} + \left(|p_1|^2 / 2p_{1,R}^3 \right) \right]^2}, \\ \Gamma_{\min 1} &= -p_{1,R}^3 16\left(a + 2bp_{1,I}\right)t - 16ap_{1,R}^3 \tau + \text{Re}\,(\xi_{10}) - 2p_{1,R}\,\text{Re}\left(\frac{c_{11}}{p_1}\right) + 1 + \sqrt{3}, \end{aligned} \tag{32}$$

$$\begin{aligned} u|_{A_3} &= u_{\min 2} = 2 \frac{-\left(1/p_{1,R}^2\right) + 2C_{11}|p_1|^2 \left(3 - 2\sqrt{3}/2p_{1,R}\right) e^{-\Gamma_{\min 2}}}{\left[C_{11} e^{-\Gamma_{\min 2}} + \left(|p_1|^2 / 2p_{1,R}^3 \right) \right]^2}, \\ \Gamma_{\min 2} &= -p_{1,R}^3 16\left(a + 2bp_{1,I}\right)t - 16ap_{1,R}^3 \tau \\ &\quad + \text{Re}\,(\xi_{10}) - 2p_{1,R}\,\text{Re}\left(\frac{c_{11}}{p_1}\right) + 1 - \sqrt{3}. \end{aligned} \tag{33}$$

Below, we investigate two interesting phenomena exhibited during the interaction between a lump and a line soliton.

(i) Fusion: we consider the case where $p_{1,R}^3 16(a + 2bp_{1,I}) > 0$. As $t \longrightarrow -\infty$, we obtain $u_{\max} \longrightarrow 64p_{1,R}^4/|p_1|^4$, $u_{\min 1} \longrightarrow -8p_{1,R}^4/|p_1|^4$, $u_{\min 2} \longrightarrow -8p_{1,R}^4/|p_1|^4$, which shows that the lump always exists before it interacts with line soliton. As $t \longrightarrow +\infty$, we have $u_{\max} \longrightarrow 0$, $u_{\min 1} \longrightarrow 0$, $u_{\min 2} \longrightarrow 0$, which indicates that the interaction between lump and line soliton results in annihilation of lump. Therefore, the fundamental semirational solution (27) describes the fusion of one lump and one line soliton in this case. We illustrate the fusion process of fundamental semirational solution (27) graphically in Figure 3. As displayed in

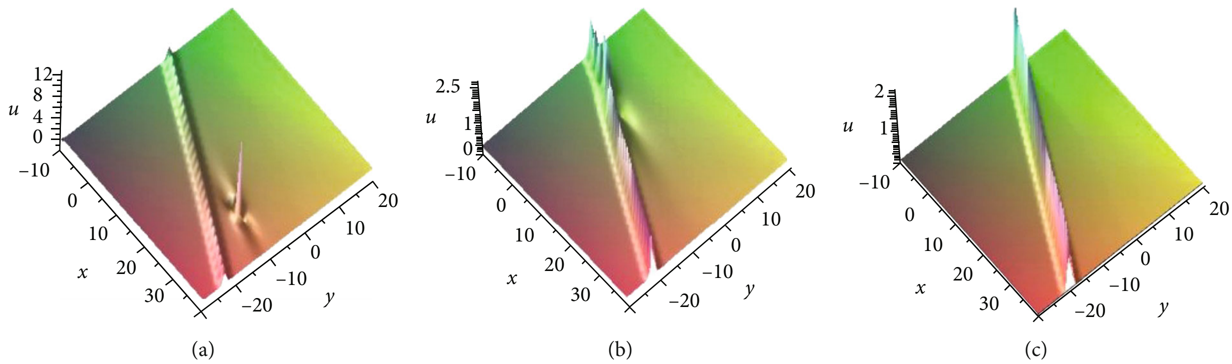


Figure 3: The fundamental semirational solution (27) with $p_1 = 1 + i$, $q_1 = 1 - i$, $a = 1$, $b = 1$, $\tau = 0$, $c_{11} = 1$, $d_{11} = 1$, $C_{11} = 1$, $\xi_{10} = \eta_{10} = 0$: (a) $t = -1/2$, (b) $t = 0$, and (c) $t = 1/2$.

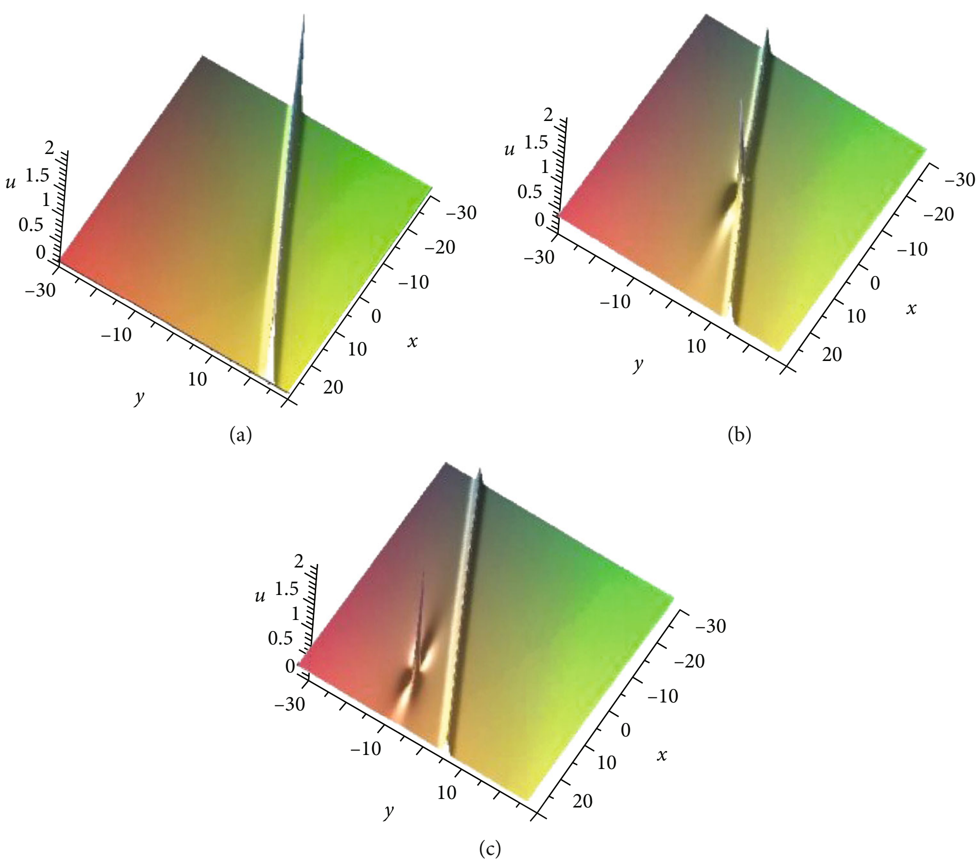


Figure 4: The fundamental semirational solution (27) with $p_1 = 1 - i$, $q_1 = 1 + i$, $a = 1$, $b = 1$, $\tau = 0$, $c_{11} = 1$, $d_{11} = 1$, $C_{11} = 1$, $\xi_{10} = \eta_{10} = 0$: (a) $t = -2$, (b) $t = 0$, and (c) $t = 2$.

Figure 3, there are a line soliton and a lump at $t = -1/2$. Then, the lump travels toward the soliton and merges with the soliton at $t = 0$. When $t = 1/2$, the lump vanishes completely and only the soliton exists

(ii) Fission: we consider the case where $p_{1,R}^3 16(a + 2b p_{1,I}) < 0$. As $t \longrightarrow -\infty$, we obtain $u_{\max} \longrightarrow 0$, $u_{\min 1} \longrightarrow 0$, $u_{\min 2} \longrightarrow 0$, which shows that the lump does not exist before it interacts with line soliton. As $t \longrightarrow +\infty$, we have $u_{\max} \longrightarrow 64 p_{1,R}^4 / |p_1|^4$, $u_{\min 1} \longrightarrow -8 p_{1,R}^4 / |p_1|^4$, $u_{\min 2} \longrightarrow -8 p_{1,R}^4 / |p_1|^4$, which indicates that the interaction between lump and line soliton results in creation of lump. Therefore, the fundamental semirational solution (27) describes the fission of one lump and one line soliton in this case. We illustrate the fission process of fundamental semirational solution (27) graphically in Figure 4. As shown in Figure 4, there is only one soliton at $t = -2$ and a lump starts to split from the soliton at $t = 0$. When $t = 2$, the lump completely separates from the soliton

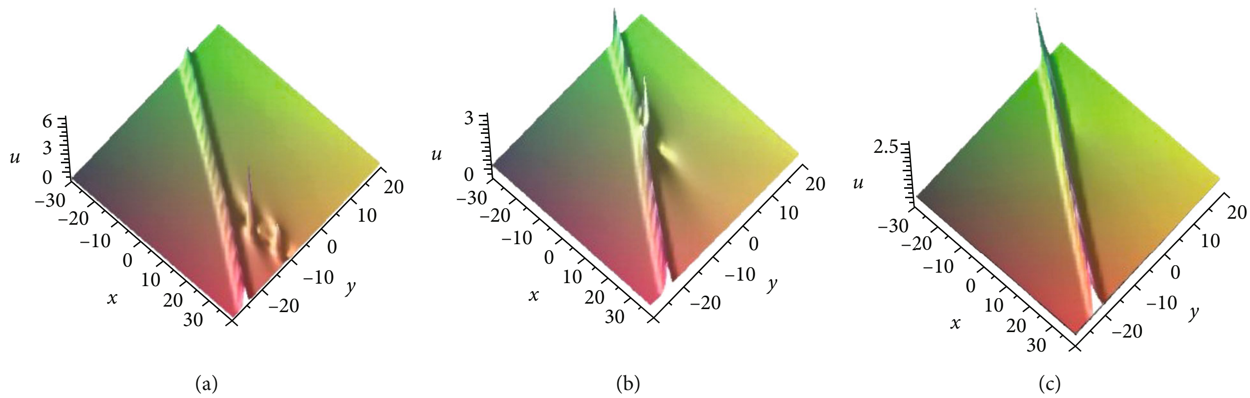


Figure 5: The high-order semirational solution $u = 2\ln(G)_{xx}$, where G is given by (34) with $p_1 = 1 + i$, $q_1 = 1 - i$, $a = 1$, $b = 1$, $\tau = 0$, $c_{12} = 1$, $d_{12} = 1$, $C_{11} = 1$, $\xi_{10} = \eta_{10} = 0$: (a) $t = -1/2$, (b) $t = 0$, and (c) $t = 1/2$.

To demonstrate the fusion and fission processes of multilumps and multiline solitons, we investigate two cases of nonfundamental semirational solutions.

Case 1. When $N = 1$, $n_1 = 2$, we obtain the high-order semirational solution for the cKP3-4 equation as follows:

$$G = e^{\xi_1+\eta_1}\left(M + c_{12}W + \frac{c_{12}d_{12}}{p_1+q_1}\right) + C_{11}, \tag{34}$$

where

$$\begin{aligned} W &= -q_1\frac{(p_1+q_1)^2 - 2q_1(p_1+q_1)}{(p_1+q_1)^4} + q_1\frac{\left(\partial_{q_1}\eta_1'\right)(p_1+q_1) - \eta_1'}{(p_1+q_1)^2} \\ &\quad + \eta_1'\left(-\frac{q_1}{(p_1+q_1)^2} + \frac{\eta_1'}{p_1+q_1}\right), \\ M &= p_1\left(\partial_{p_1}W + p_1\partial_{p_1^2}W - \frac{d_{12}(p_1+q_1)^2 - 2p_1d_{12}(p_1+q_1)}{(p_1+q_1)^4}\right. \\ &\quad \left. + \left(\partial_{p_1}\xi_1'\right)W + \xi_1'\partial_{p_1}W + \frac{d_{12}\left(\partial_{p_1}\xi_1'\right)(p_1+q_1) - d_{12}\xi_1'}{(p_1+q_1)^2}\right) \\ &\quad + \xi_1'\left(p_1\partial_{p_1}W - \frac{p_1d_{12}}{(p_1+q_1)^2} + \xi_1'W + \frac{\xi_1'd_{12}}{p_1+q_1}\right), \end{aligned} \tag{35}$$

in which we have taken $c_{10} = d_{10} = 1$, $c_{11} = d_{11} = 0$. The high-order semirational solution (34) describes the resonant interaction between one soliton and two lumps. We demonstrate the fusion and fission processes of high-order semirational solution (34) graphically in Figures 5 and 6, respectively. As shown in Figure 5, the two lumps immerse into the line soliton, so we observe only one line soliton in Figure 5(c). In Figure 6, there is only one line soliton at $t = -2$. As time progresses, two lumps arise from the line soliton and then the two lumps separate completely from the line soliton.

Case 2. When $N = 2$, $n_r = 1$ $(r = 1, 2)$, we obtain the multiple semirational solution for the cKP3-4 equation which is expressed as

$$G = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \tag{36}$$

in which

$$\begin{aligned} a_{11} &= C_{11} + e^{(\xi_1+\eta_1)}\left[\left(\xi_1' + c_{11} - \frac{p_1}{p_1+q_1}\right)\right. \\ &\quad \left.\cdot\left(\eta_1' + d_{11} - \frac{q_1}{p_1+q_1}\right) + \frac{p_1q_1}{(p_1+q_1)^2}\right]\frac{1}{p_1+q_1}, \\ a_{12} &= C_{12} + e^{(\xi_1+\eta_2)}\left[\left(\xi_1' + c_{11} - \frac{p_1}{p_1+q_2}\right)\right. \\ &\quad \left.\cdot\left(\eta_2' + d_{21} - \frac{q_2}{p_1+q_2}\right) + \frac{p_1q_2}{(p_1+q_2)^2}\right]\frac{1}{p_1+q_2}, \\ a_{21} &= C_{21} + e^{(\xi_2+\eta_1)}\left[\left(\xi_2' + c_{21} - \frac{p_2}{p_2+q_1}\right)\right. \\ &\quad \left.\cdot\left(\eta_1' + d_{11} - \frac{q_1}{p_2+q_1}\right) + \frac{p_2q_1}{(p_2+q_1)^2}\right]\frac{1}{p_2+q_1}, \\ a_{22} &= C_{22} + e^{(\xi_2+\eta_2)}\left[\left(\xi_2' + c_{21} - \frac{p_2}{p_2+q_2}\right)\right. \\ &\quad \left.\cdot\left(\eta_2' + d_{21} - \frac{q_2}{p_2+q_2}\right) + \frac{p_2q_2}{(p_2+q_2)^2}\right]\frac{1}{p_2+q_2}, \end{aligned} \tag{37}$$

where we have taken $c_{10} = 1$, $d_{10} = 1$. The multiple semirational solution (36) describes resonant interaction between two lumps and two line solitons. We illustrate the fusion and fission processes of multiple semirational solution (36) graphically in Figures 7 and 8, respectively. As displayed in Figure 7, the two lumps approach two intersecting solitons and eventually fuse into the solitons. As shown in Figure 8,

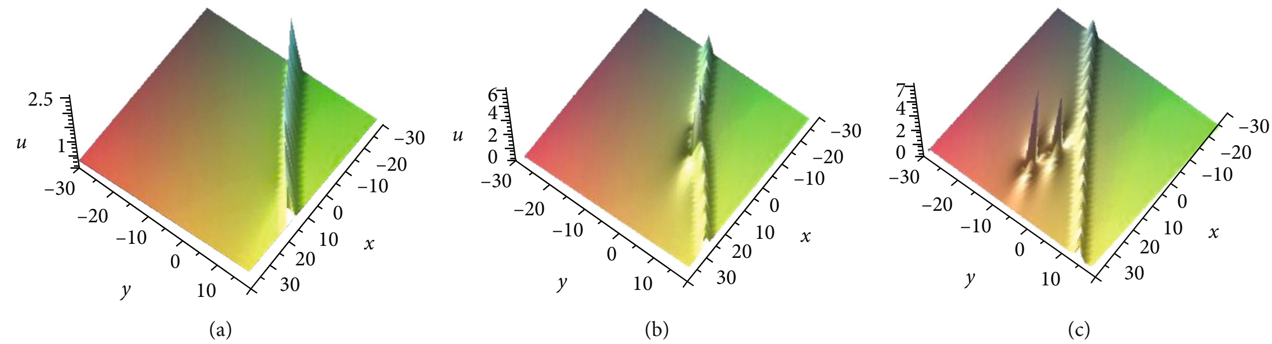


FIGURE 6: The high-order semirational solution $u = 2\ln(G)_{xx}$, where G is given by (34) with $p_1 = 1 - i$, $q_1 = 1 + i$, $a = 1$, $b = 1$, $\tau = 0$, $c_{12} = 1$, $d_{12} = 1$, $C_{11} = 1$, $\xi_{10} = \eta_{10} = 0$: (a) $t = -2$, (b) $t = -0.2$, and (c) $t = 2$.

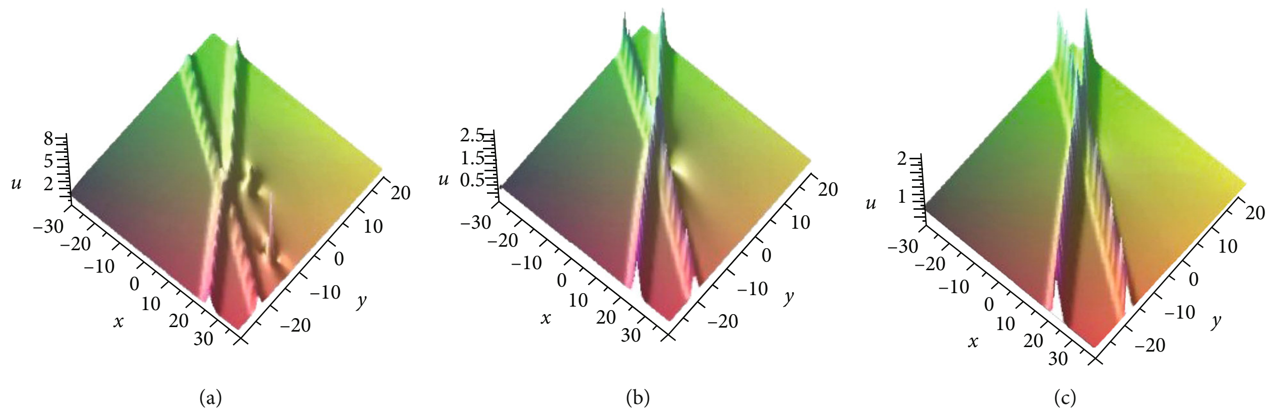


FIGURE 7: The multiple semirational solution $u = 2\ln(G)_{xx}$, where G is given by (36) with $p_1 = 1 + i$, $q_1 = 1 - i$, $p_2 = 1 + (1/2)i$, $q_2 = 1 - (1/2)i$, $a = 1$, $b = 1$, $\tau = 0$, $c_{11} = 1$, $d_{11} = 1$, $c_{21} = 1 + i$, $d_{21} = 1 - i$, $C_{11} = 1$, $C_{12} = C_{21} = 0$, $C_{22} = 2$, $\xi_{10} = \eta_{10} = \xi_{20} = \eta_{20} = 0$: (a) $t = -1/2$, (b) $t = 0$, and (c) $t = 1/2$.

the two lumps arise from the two intersecting solitons and then separate from the solitons.

In order to demonstrate the rogue lump phenomenon, we take $N = 2$ and $n_r = 2 - r$, $n_s = 2 - s$ $(r, s = 1, 2)$ in semirational solution (25) and impose some parameter constraints. The semirational solution (25) with $N = 2$ and $n_r = 2 - r$, $n_s = 2 - s$ $(r, s = 1, 2)$ can be expressed as

$$G = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \tag{38}$$

where

$$\begin{aligned}
a_{11} &= C_{11} + e^{(\xi_1+\eta_1)}\left[\left(\xi_1' - \frac{p_1}{p_1+q_1}\right)\left(\eta_1' - \frac{q_1}{p_1+q_1}\right) + \frac{p_1 q_1}{(p_1+q_1)^2}\right]\frac{1}{p_1+q_1},\\
a_{12} &= C_{12} + e^{(\xi_1+\eta_2)}\left(\xi_1' - \frac{p_1}{p_1+q_2}\right)\frac{1}{p_1+q_2},\\
a_{21} &= C_{21} + e^{(\xi_2+\eta_1)}\left(\eta_1' - \frac{q_1}{p_2+q_1}\right)\frac{1}{p_2+q_1},\\
a_{22} &= C_{22} + e^{(\xi_2+\eta_2)}\frac{1}{p_2+q_2},
\end{aligned} \tag{39}$$

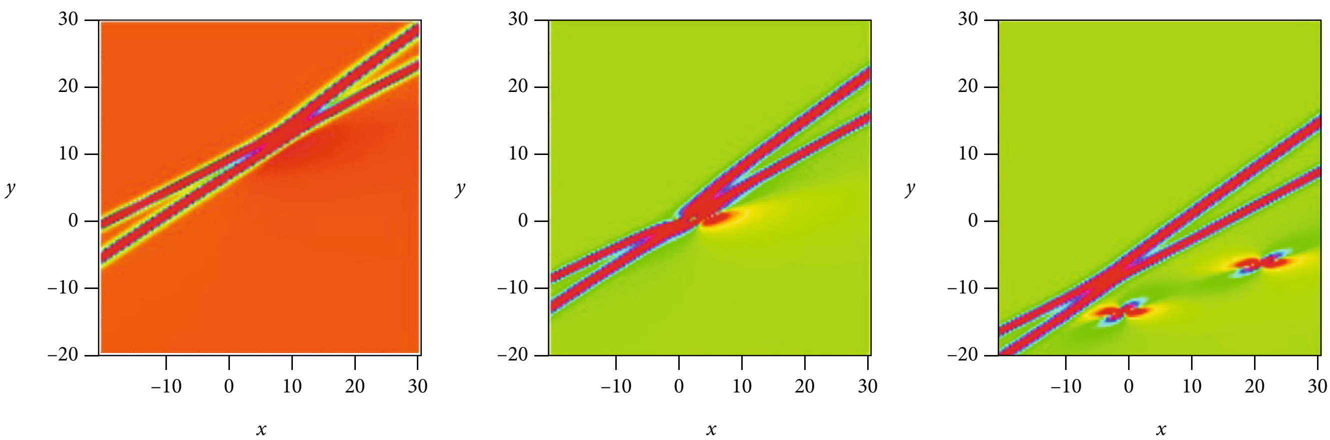
in which we have taken $c_{10} = d_{10} = 1$, $c_{11} = d_{11} = 0$. Furthermore, if we take $p_r = p$, $q_s = p^*$, $C_{11} = C_{22} = 1$, $C_{12} = C_{21} = 0$, $\xi_{r0} = \eta_{r0}$ $(r, s = 1, 2)$ in (38), then we obtain

$$\begin{aligned}
G = {} & 1 + e^{2(\xi_R+\xi_{10})}\left(\left|\xi' - \frac{1}{2p_R}\right|^2 + \frac{1}{4p_R^2} + e^{2(\xi_{20}-\xi_{10})}\frac{1}{|p|^2}\right)\frac{|p|^2}{2p_R}\\
& + e^{4\xi_R+2(\xi_{10}+\xi_{20})}\frac{|p|^2}{16p_R^4},
\end{aligned} \tag{40}$$

where $\xi = px - ip^2y + (4ap^3 - 2bip^4)t + 4ap^3\tau$, $\xi' = x - 2ipy + (12ap^2 - 8bip^3)t + 12ap^2\tau$. Noticing the dependent variable transformation $u = 2\ln(G)_{xx}$, we rewrite G in (40) as follows:

$$\begin{aligned}
\widehat{G} = {} & e^{-2(\xi_R+\xi_{10})}\frac{1}{|p|^2} + \left(\left|\xi' - \frac{1}{2p_R}\right|^2 + \frac{1}{4p_R^2} + e^{2(\xi_{20}-\xi_{10})}\frac{1}{|p|^2}\right)\frac{1}{2p_R}\\
& + e^{2(\xi_R+\xi_{20})}\frac{1}{16p_R^4}.
\end{aligned} \tag{41}$$

The semirational solution (41) describes the resonant collision between one lump and two solitons, in which the



(a) (b) (c)

FIGURE 8: The multiple semirational solution $u = 2\ln(G)_{xx}$, where G is given by (36) with $p_1 = 1 - i$, $q_1 = 1 + i$, $p_2 = 1 - (3/4)i$, $q_2 = 1 + (3/4)i$, $a = 1$, $b = 1$, $\tau = 0$, $c_{11} = 1$, $d_{11} = 1$, $c_{21} = 1 + i$, $d_{21} = 1 - i$, $C_{11} = 1$, $C_{12} = C_{21} = 0$, $C_{22} = 2$, $\xi_{10} = \eta_{10} = \xi_{20} = \eta_{20} = 0$: (a) $t = -2$, (b) $t = 0$, and (c) $t = 2$.

peak of lump moves along the trajectory $[x(t), y(t)]$:

$$\xi_R' - \frac{1}{2p_R} = 0, \xi_I' = 0, \quad (42)$$

where $\xi_R{}' = x + 2p_I y + [12a(p_R^2 - p_I^2) + 8b(3p_R^2 p_I - p_I^3)]t + 12a(p_R^2 - p_I^2)\tau$ and $\xi_I{}' = -2p_R y + [24ap_R p_I - 8b(p_R^3 - 3p_R p_I^2)]t + 24ap_R p_I \tau$. The trajectories of two solitons also can be calculated from equation (41). Soliton 1 moves along

$$L_1 = 0. \quad (43)$$

Soliton 2 moves along

$$L_2 = 0, \quad (44)$$

where

$$\begin{aligned} L_1 &= 2(\xi_R + \xi_{10}) + L = 2(p_R x + 2p_R p_I y \\ &\quad + \left[4a\left(p_R^3 - 3p_R p_I^2\right) + 2b\left(4p_R^3 p_I - 4p_R p_I^3\right)\right]t \\ &\quad + 4a\left(p_R^3 - 3p_R p_I^2\right)\tau + \xi_{10}) + L, \\ L_2 &= 2(\xi_R + \xi_{20}) - L = 2(p_R x + 2p_R p_I y \\ &\quad + \left[4a\left(p_R^3 - 3p_R p_I^2\right) + 2b\left(4p_R^3 p_I - 4p_R p_I^3\right)\right]t \\ &\quad + 4a\left(p_R^3 - 3p_R p_I^2\right)\tau + \xi_{20}) - L, \\ L &= \ln\left[\left(\left|\xi' - \frac{1}{2p_R}\right|^2 + \frac{1}{4p_R^2} + e^{2(\xi_{20} - \xi_{10})}\frac{1}{|p|^2}\right)\frac{1}{2p_R}\right]. \end{aligned} \quad (45)$$

For the asymptotic analysis of lump, when $\xi_R' - 1/2p_R \approx 0, \xi_I' \approx 0$, we obtain $\xi_R \longrightarrow \pm\infty$ as $t \longrightarrow \pm\infty$ by applying the relation $\xi_R = p_R \xi_R' + (8a - 16p_I)p_R^3 t + 8ap_R^3\tau$. Thus, we derive the asymptotic form of the lump denoted as G_{lump} from solution (40) as follows:

$$G_{\text{lump}} \sim 1, \text{ as } t \longrightarrow \pm\infty. \quad (46)$$

Substituting G_{lump} into $u_{\text{lump}} = 2\ln(G_{\text{lump}})_{xx}$ gives

$$u_{\text{lump}} \sim 0, \text{ as } t \longrightarrow \pm\infty. \quad (47)$$

For the asymptotic analysis of soliton 1, when $L_1 \approx 0$, the semirational solution (40) can be expressed as

$$G = 1 + |p|^2 e^{L_1} + \frac{|p|^2}{16p_R^4 e^{2L}} e^{2L_1 + 2(\xi_{20} - \xi_{10})}, \quad (48)$$

and $e^L \longrightarrow +\infty$ when $t \longrightarrow \pm\infty$. Thus, we obtain the following asymptotic form of soliton 1 which is denoted as G_{soliton1} from (48):

$$\begin{gathered} G_{\text{soliton1}} \sim 1 + |p|^2 e^{L_1} = 1 + |p|^2 e^{2(\xi_R + \xi_{10}) + L}, \text{ as } t \longrightarrow \pm\infty, \\ u_{\text{soliton1}} \sim 2\ln(G_{\text{soliton1}})_{xx}, \text{ as } t \longrightarrow \pm\infty. \end{gathered} \quad (49)$$

For the asymptotic analysis of soliton 2, when $L_2 \approx 0$, the semirational solution (40) can be expressed as

$$G = 1 + |p|^2 e^{L_2 + 2(\xi_{10} - \xi_{20}) + 2L} + \frac{|p|^2}{16p_R^4} e^{2L_2 + 2(\xi_{10} - \xi_{20}) + 2L}, \quad (50)$$

and $e^L \longrightarrow +\infty$ when $t \longrightarrow \pm\infty$. Noticing the dependent variable transformation $u = 2\ln(G)_{xx}$, equation (50) can be written as

$$G = e^{-2L - 2(\xi_{10} - \xi_{20})} + |p|^2 e^{L_2} + \frac{|p|^2}{16p_R^4} e^{2L_2}. \quad (51)$$

Therefore, we obtain the following asymptotic form of soliton 2 which is denoted as G_{soliton2} from (51):

$$G_{\text{soliton2}} \sim e^{L_2}|p|^2\left(1+e^{L_2-\ln\left(16p_R^4\right)}\right), \text{ as } t\longrightarrow \pm\infty,$$
$$u_{\text{soliton2}} \sim 2\ln\left(G_{\text{soliton2}}\right)_{xx}, \text{ as } t\longrightarrow \pm\infty. \tag{52}$$

We conclude from above asymptotic analysis that the lump exists for a finite period of time, and the solitons do not change their velocities and shapes before and after the collision. Therefore, the lump is localized in time as well as the two spatial dimensions and exhibits rogue wave phenomenon. We illustrate the rogue lump in Figure 9. In Figure 9(a), there are only two line solitons. Then, the lump starts to detach from one (soliton 1) of them, as shown in Figure 9(b). Figures 9(c)–9(e) depict the lump moves toward another soliton (soliton 2). Subsequently, the lump begins to fuse into soliton 2, as shown in Figure 9(f). In Figure 9(g), the lump completely merges with soliton 2.

4. The cKP3-4 Equation with Self-Consistent Sources

In this section, we construct the cKP3-4 equation with self-consistent sources by applying the source generation procedure. Furthermore, we present the Wronskian solution for the cKP3-4 equation with self-consistent sources and the cKP3-4 equation.

In order to construct the cKP3-4 equation with self-consistent sources, we change $C_{ss}(s=1,\cdots,N)$ in (18) to the following form:

$$C_{ss}=\begin{cases} C_s(t), & 1\le s\le K\le N,\\ C_{ss}, & K<s\le N,\end{cases} \tag{53}$$

where $C_s(t)(1\le s\le K\le N)$ are arbitrary functions of time t. The changed G still satisfies equation (4), but it does not satisfy equation (5). Therefore, we introduce other new functions defined by

$$d_s=\sqrt{2\dot{C}_s(t)}\begin{vmatrix} & & & & & C_{1,s}+\int f_1 g_s d_x\\ & & & & & \vdots\\ & & & & & C_{s-1,s}+\int f_{s-1} g_s d_x\\ & & G_{ss} & & & C_{s+1,s}+\int f_{s+1} g_s d_x\\ & & & & & \vdots\\ & & & & & C_{N,s}+\int f_N g_s d_x\\ -g_1 & \cdots & -g_{s-1} & -g_{s+1} & \cdots\ -g_N & -g_s\end{vmatrix}, \tag{54}$$

$$h_s=\sqrt{2\dot{C}_s(t)}\begin{vmatrix} & & & & & f_1\\ & & & & & \vdots\\ & & & & & f_{s-1}\\ & & G_{ss} & & & f_{s+1}\\ & & & & & \vdots\\ & & & & & f_N\\ C_{s,1}+\int f_s g_1 d_x & \cdots & C_{s,s-1}+\int f_s g_{s-1} d_x & C_{s,s+1}+\int f_s g_{s+1} d_x & \cdots\ C_{s,N}+\int f_s g_N d_x & f_s\end{vmatrix}, \tag{55}$$

where $G_{rs}(r,s=1,\cdots,N)$ is the cofactor of element a_{rs}. We can show that these new functions $G, d_s, h_s(s=1,\cdots,K\le N)$ satisfy the following bilinear equations:

$$\left[a\left(2bD_x^3D_y-3D_xD_t+3D_xD_\tau\right)+bDyD_\tau\right]G\cdot G=3a\sum_{s=1}^{K}d_s\cdot h_s, \tag{56}$$

$$\left(D_x^2-iD_y\right)G\cdot d_s=0, s=1,\cdots K, \tag{57}$$

$$\left(D_x^2-iD_y\right)h_s\cdot G=0, s=1,\cdots K. \tag{58}$$

Equations (4) and (56)–(58) constitute the bilinear cKP3-4 equation with self-consistent sources. Through the dependent variable transformations

$$u=2\ln(G)_{xx}, \Phi_s=\frac{d_s}{G}, \Psi_s=\frac{h_s}{G}(s=1,\cdots,K), \tag{59}$$

the bilinear equations (4) and (56)–(58) are transformed

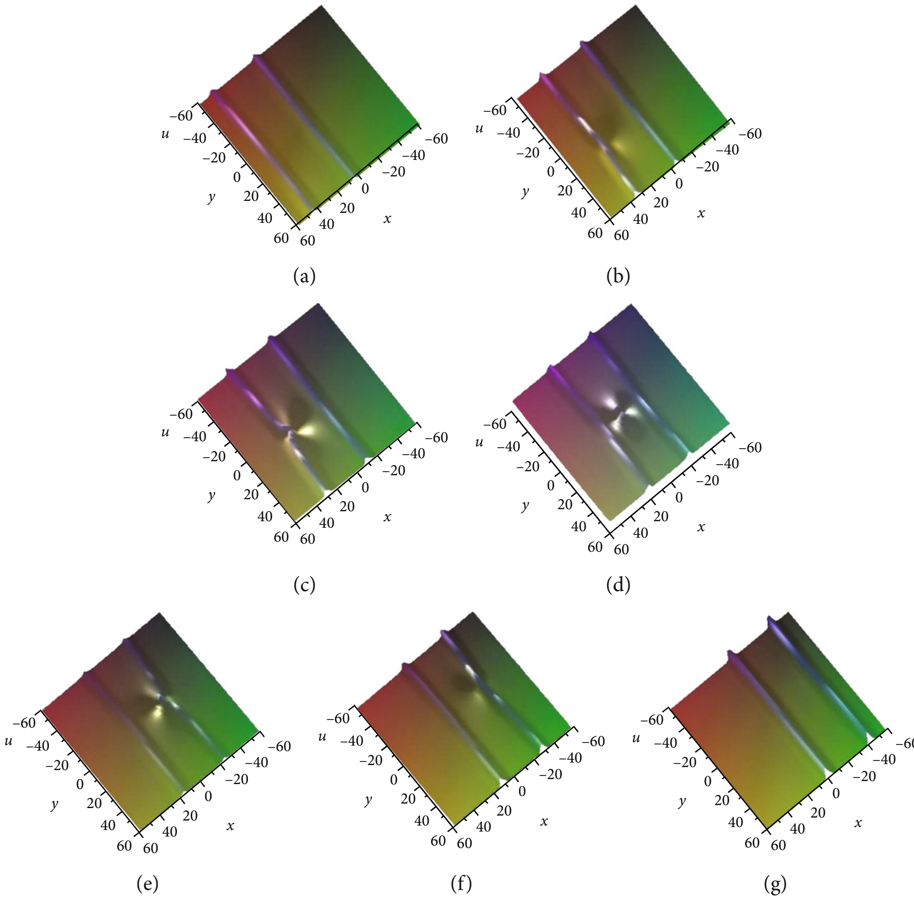


Figure 9: The semirational solution $u = 2\ln(G)_{xx}$, where G is given by (40) with $p_1 = 1/2$, $q_1 = 1/2$, $p_2 = 1/2$, $q_2 = 1/2$, $c_{10} = d_{10} = 1$, $c_{11} = d_{11} = 0$, $a = 1$, $b = 1$, $\xi_{10} = \eta_{10} = 2\pi$, $\xi_{20} = \eta_{20} = -2\pi$, $C_{11} = C_{22} = 1$, $C_{12} = C_{21} = 0$: (a) $t = -25$, (b) $t = -12$, (c) $t = -6$, (d) $t = 0$, (e) $t = 6$, (f) $t = 12$, and (g) $t = 25$.

into the nonlinear cKP3-4 equation with self-consistent sources:

$$a(6uu_x + u_{xxx} - 3w_y) + b(2wu_x - z_y + u_{xxy} + 4uu_y) - u_t = \sum_{s=1}^{K} (\Psi_s \Phi_s)_x, \quad (60)$$

$$\Phi_{s,xx} + u\Phi_s + i\Phi_{s,y} = 0, \quad s = 1, \cdots K, \quad (61)$$

$$\Psi_{s,xx} + u\Psi_s - i\Psi_{s,y} = 0, \quad s = 1, \cdots K. \quad (62)$$

As an application of the Grammian solutions (18), (54), and (55) for the cKP3-4 equation with self-consistent sources, we obtain its N-soliton solution by taking

$$C_s(t) = \frac{e^{(2a_s(t))}}{p_s + q_s}, s = 1, \cdots, K \le N,$$
$$C_{ss} = \frac{1}{p_s + q_s}, s = k + 1, \cdots, N,$$
$$C_{rs} = 0, r \ne s(r, s = 1,\cdots,N),$$

$$f_r = e^{\xi_r}, \xi_r = p_r x - ip_r^2 y + (4ap_r^3 - 2bip_r^4)t + 4ap_r^3\tau, r = 1, \cdots, N,$$

$$g_s = e^{\eta_s}, \eta_s = q_s x + iq_s^2 y + (4aq_s^3 + 2biq_s^4)t + 4aq_s^3\tau, s = 1, \cdots, N, \quad (63)$$

where $a_s(t)(s = 1,\cdots,K \le N)$ are arbitrary functions of t satisfying $\dot{a}_s(t) \ge 0$ for arbitrary t. When $N = 1, K = 1$, we obtain the following 1-soliton solution for the cKP3-4 equation with self-consistent sources

$$u = 2\ln\left(1 + e^{(\xi_1 + \eta_1 - 2a_1(t))}\right)_{xx},$$
$$\phi_1 = -2\sqrt{\dot{a}_1(t)(p_1 + q_1)}\frac{e^{(\eta_1 - a_1(t))}}{1 + e^{(\xi_1 + \eta_1 - 2a_1(t))}}, \quad (64)$$
$$\psi_1 = 2\sqrt{\dot{a}_1(t)(p_1 + q_1)}\frac{e^{(\xi_1 - a_1(t))}}{1 + e^{(\xi_1 + \eta_1 - 2a_1(t))}}.$$

When $N = 2, K = 1$, the 2-soliton solution for the cKP3-4 with self-consistent sources is expressed as

$$u = 2\ln\left(1 + e^{(\xi_1+\eta_1-2a_1(t))} + e^{(\xi_2+\eta_2)} + A_{12}e^{(\xi_1+\eta_1+\xi_2+\eta_2-2a_1(t))}\right)_{xx},$$
$$\phi_1 = -\frac{2\sqrt{\dot a_1(t)(p_1+q_1)}\left(1 + a_1 e^{(\eta_1-a_1(t))}\right)}{1 + e^{(\xi_1+\eta_1-2a_1(t))} + e^{(\xi_2+\eta_2)} + A_{12}e^{(\xi_1+\eta_1+\xi_2+\eta_2-2a_1(t))}},$$
$$\psi_1 = \frac{2\sqrt{\dot a_1(t)(p_1+q_1)}\left(1 + b_1 e^{(\xi_1-a_1(t))}\right)}{1 + e^{(\xi_1+\eta_1-2a_1(t))} + e^{(\xi_2+\eta_2)} + A_{12}e^{(\xi_1+\eta_1+\xi_2+\eta_2-2a_1(t))}}, \tag{65}$$

where $A_{12} = (p_1-p_2)(q_1-q_2)/(p_1+q_2)(p_2+q_1)$, $a_1 = (q_1 - q_2)/(p_2+q_1)$, $b_1 = (p_1-p_2)/(p_1+q_2)$.

Proposition 3. *The cKP3-4 equation with self-consistent sources (4) and (56)–(58) has the following Wronskian solution:*

$$G = (d_0,\cdots,d_{N-1}, N,\cdots,1), \tag{66}$$

$$h_s = -\sqrt{2\dot\gamma_s(t)}(d_0,\cdots,d_N, N,\cdots,1,\alpha_s), \quad s = 1,\cdots,K, \tag{67}$$

$$d_s = \sqrt{2\dot\gamma_s(t)}(d_0,\cdots,d_{N-2}, N,\cdots,\widehat{s},\cdots,1), \quad s = 1,\cdots,K, \tag{68}$$

where the Pfaffian elements are defined by

$$(d_m, r) = f_r^{(m)}, (d_m,\alpha_r) = \rho_{r2}^{(m)}, (d_m, d_l) = (r,s) = 0, (\alpha_r, s) = (\alpha_r,\alpha_s) = 0 \quad (r,s = 1,\cdots,N), \tag{69}$$

in which m, l are integers, $f_r = \rho_{r1} + (-1)^{r-1}C_r(t)\rho_{r2}(r = 1,\cdots,N)$ where ρ_{r1}, ρ_{r2} $(r = 1,\cdots,N)$ are functions of x, y, t and

$$C_r(t) = \begin{cases} \gamma_r(t), & r = 1,\cdots,K,\\ c_r, & r = K+1,\cdots,N, \end{cases} \tag{70}$$

with $K \le N$, K is a positive integer. Here, $\gamma_r(t)(r = 1,\cdots,K)$ are arbitrary functions of t, $c_r(r = K+1,\cdots,N)$ are arbitrary constants, and ρ_{r1}, ρ_{r2} satisfy the following relations:

$$\frac{\partial\rho_{r1}}{\partial y} = -i\frac{\partial^2\rho_{r1}}{\partial x^2}, \frac{\partial\rho_{r1}}{\partial\tau} = 4a\frac{\partial^3\rho_{r1}}{\partial x^3}, \frac{\partial\rho_{r1}}{\partial t} = 4a\frac{\partial^3\rho_{r1}}{\partial x^3} - 2bi\frac{\partial^4\rho_{r1}}{\partial x^4}, \tag{71}$$

$$\frac{\partial\rho_{r2}}{\partial y} = -i\frac{\partial^2\rho_{r2}}{\partial x^2}, \frac{\partial\rho_{r2}}{\partial\tau} = 4a\frac{\partial^3\rho_{r2}}{\partial x^3}, \frac{\partial\rho_{r2}}{\partial t} = 4a\frac{\partial^3\rho_{r2}}{\partial x^3} - 2bi\frac{\partial^4\rho_{r2}}{\partial x^4}. \tag{72}$$

Proposition 3 can be proved by using Wronskian technique. For example, applying the dispersion relations (71) and (72), the derivatives of Wronskian G can be expressed in the form of Wronskians:

$$\begin{aligned}
G_{xxxx} &= (d_0,\cdots,d_{N-5}, d_{N-3}, d_{N-2}, d_{N-1}, d_N, N,\cdots,1)\\
&\quad + (d_0,\cdots,d_{N-2}, d_{N+3}, N,\cdots,1)\\
&\quad + 3(d_0,\cdots,d_{N-3}, d_{N-1}, d_{N+2}, N,\cdots,1)\\
&\quad + 2(d_0,\cdots,d_{N-3}, d_N, d_{N+1}, N,\cdots,1)\\
&\quad + 3(d_0,\cdots,d_{N-4}, d_{N-2}, d_{N-1}, d_{N+1}, N,\cdots,1),\\
G_t &= 4a[(d_0,\cdots,d_{N-4}, d_N, d_{N-2}, d_{N-1}, N,\cdots,1)\\
&\quad + (d_0,\cdots,d_{N-3}, d_{N+1}, d_{N-1}, N,\cdots,1)\\
&\quad + (d_0,\cdots,d_{N-2}, d_{N+2}, N,\cdots,1)]\\
&\quad - 2bi[(d_0,\cdots,d_{N-5}, d_N, d_{N-3}, d_{N-2}, d_{N-1}, ,N,\cdots,1)\\
&\quad + (d_0,\cdots,d_{N-4}, d_{N+1}, d_{N-2}, d_{N-1}, N,\cdots,1)\\
&\quad + (d_0,\cdots,d_{N-3}, d_{N+2}, d_{N-1}, N,\cdots,1)\\
&\quad + (d_0,\cdots,d_{N-2}, d_{N+3}, N,\cdots,1)]\\
&\quad + \sum_{s=1}^{K}\dot\gamma_s(t)(d_0,\cdots,d_{N-1}, N,\cdots,\widehat{s},\cdots,1,\alpha_s),
\end{aligned} \tag{73}$$

and equation (56) can be reduced to the following Plücker relation for determinants:

$$\begin{aligned}
&24abi[-(d_0,\cdots,d_{N-4}, d_N, d_{N+1}, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-3}, d_{N-1}, d_N, N,\cdots,1)\\
&-(d_0,\cdots,d_{N-3}, d_N, d_{N+2}, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-2}, d_{N-1}, N,\cdots,1)\\
&+(d_0,\cdots,d_{N-3}, d_{N-1}, d_{N+2}, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-2}, d_N, N,\cdots,1)\\
&+(d_0,\cdots,d_{N-4}, d_{N-1}, d_{N+1}, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-3}, d_N, N,\cdots,1)\\
&-(d_0,\cdots,d_{N-2}, d_{N+2}, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-3}, d_{N-1}, d_N, N,\cdots,1)\\
&-(d_0,\cdots,d_{N-4}, d_{N-1}, d_N, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-3}, d_{N+1}, N,\cdots,1)]\\
&-6a\sum_{s=1}^{K}\dot\gamma_s(t)[(d_0,\cdots,d_{N-2}, d_N, N,\cdots,\widehat{s},\cdots,1,\alpha_s)\\
&\cdot(d_0,\cdots,d_{N-1}, N,\cdots,1) - (d_0,\cdots,d_{N-2}, d_N, N,\cdots,1)\\
&\cdot(d_0,\cdots,d_{N-1}, N,\cdots,\widehat{s},\cdots,1,\alpha_s)\\
&+(d_0,\cdots,d_N, N,\cdots,1,\alpha_s)\\
&\cdot(d_0,\cdots,d_{N-2}, N,\cdots,\widehat{s},\cdots,1)] = 0.
\end{aligned} \tag{74}$$

Similarly, we can show that (66)–(68) is a solution to equations (4), (57), and (58). By taking $\gamma_r(t) = c_r(r = 1,\cdots, K)$ (c_r is a constant) in the Wronskian solution (66)–(68), we obtain the Wronskian solution for cKP3-4 equations (4) and (5).

5. Conclusion

In this paper, we apply the Hirota bilinear method and determinant technique to derive the M-lump solution in terms of Matsuno determinant for the cKP3-4 equation. Furthermore, we obtain the semirational solution for the cKP3-4 equation from its Grammian solution and illustrate the dynamical properties of the semirational solution. The asymptotic analysis of the semirational solutions shows that they describe fusion and fission processes of lumps and line solitons and rogue lump phenomena. It is interesting for us to further study the multirogue lump phenomena for the cKP3-4 equation by investigating its higher-order semirational solutions. In addition, we construct the cKP3-4 equation with self-consistent sources via the source generation procedure and present its Grammian and Wronskian solution. As an application of the Grammian solution, we derive the N-soliton solution of the cKP3-4 equation with self-consistent sources. If we take the special case $a=-1, b=0$ in equations (60)–(62), we get

$$\begin{aligned} &6uu_x + u_{xxx} - 3\int u_{yy}dx + u_t = -\sum_{s=1}^{K} (\Psi_s\Phi_s)_x, \\ &\Phi_{s,xx} + u\Phi_s + i\Phi_{s,y} = 0, s = 1, \cdots K, \\ &\Psi_{s,xx} + u\Psi_s - i\Psi_{s,y} = 0, s = 1, \cdots K, \end{aligned} \tag{75}$$

which is the KP1 equation with self-consistent sources given in Ref. [45]. And by taking $a=0, b=1$ in equations (60)–(62), we obtain

$$\begin{aligned} &6u_y u_x + 2u_{xx}\int u_y dx - \int u_{yyy}dx + u_{xxxy} \\ &\quad + 4uu_{xy} - u_{xt} = \sum_{s=1}^{K} (\Psi_s\Phi_s)_{xx}, \\ &\Phi_{s,xx} + u\Phi_s + i\Phi_{s,y} = 0, s = 1, \cdots K, \\ &\Psi_{s,xx} + u\Psi_s - i\Psi_{s,y} = 0, s = 1, \cdots K, \end{aligned} \tag{76}$$

which is the DJKM equation with self-consistent sources [56]. The lump and rogue wave solution for the KP equation with self-consistent sources is derived in [57, 58]. It is of interest for us to further investigate the rational solution and semirational solution of the cKP3-4 equation with self-consistent sources and their dynamical properties.

Appendix

A. Proof of Proposition 1

In this appendix, we prove that the Nth-order rational solution (6) given in Proposition 1 satisfies cKP3-4 equations (4) and (5) applying double-sum identity. For computational convenience, we define matrix B:

$$B = (b_{rs})|_{1\le r,s\le N}, \tag{A.1}$$

where

$$b_{rs} = \begin{cases} \dfrac{1}{2i}\theta_r, & \text{for } r = s, \\ \dfrac{1}{p_r - p_s}, & \text{for } r \ne s. \end{cases} \tag{A.2}$$

The determinant $|B|$ for the matrix B is the Matsuno determinant [54]. Applying the double-sum identity [54]:

$$\sum_{r=1}^{N}\sum_{s=1}^{N}(f_r + g_s)a_{rs}A_{rs} = \sum_{r=1}^{N}(f_r + g_r)a_{rr}A_{rr}, \tag{A.3}$$

where $A_{rs}(r, s = 1,\cdots,N)$ is the cofactor of the element a_{rs} in an arbitrary determinant $A = \det |a_{rs}|_{1\le r,s\le N}$; the following identities of cofactors for the Matsuno determinant $|B|$ can be derived [54]:

$$\sum_{r=1}^{N}\sum_{s=1}^{N} B_{rs} = \sum_{r=1}^{N} B_{rr}, \tag{A.4}$$

$$\sum_{r=1}^{N}\sum_{s=1}^{N}(p_r + p_s)B_{rs} = 2\sum_{r=1}^{N} p_r B_{rr}, \tag{A.5}$$

$$\sum_{r=1}^{N}\sum_{s=1}^{N}\left(p_r^2 + p_r p_s + p_s^2\right)B_{rs} = 3\sum_{r=1}^{N} p_r^2 B_{rr}, \tag{A.6}$$

$$\sum_{r=1}^{N}\sum_{s=1}^{N}\left(p_r^3 + p_r p_s^2 + p_r^2 p_s + p_s^3\right)B_{rs} = 4\sum_{r=1}^{N} p_r^3 B_{rr}, \tag{A.7}$$

where $B_{rs}(r, s = 1,\cdots,N)$ is the cofactor of the element b_{rs} in Matsuno determinant $|B|$. We can derive the following differential formula for the determinant F by applying (A.4)–(A.7):

$$F = (2i)^N |B|, F_x = (2i)^{N-1}\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix}, \tag{A.8}$$

$$F_{xx} = (2i)^{N-2}\left(\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ p_1 & \cdots & p_N & 0 \end{vmatrix} + \begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -1 & \cdots & -1 & 0 \end{vmatrix}\right), \tag{A.9}$$

$$F_y = \frac{(2i)^{N-1}}{2}\left(\begin{vmatrix} & & & -1 \\ & B & & \vdots \\ & & & -1 \\ p_1 & \cdots & p_N & 0 \end{vmatrix} + \begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -1 & \cdots & -1 & 0 \end{vmatrix}\right), \tag{A.10}$$

$$F_{xy} = \frac{(2i)^{N-2}}{2}\left(\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ p_1^2 & \cdots & p_N^2 & 0 \end{vmatrix} + \begin{vmatrix} & & & p_1^2 \\ & B & & \vdots \\ & & & p_N^2 \\ -1 & \cdots & -1 & 0 \end{vmatrix}\right), \tag{A.11}$$

$$F_{x\tau} + 3aF_{yy} - aF_{xxxx} = -3a\cdot(2i)^{N-2}\begin{vmatrix} & & & 1 & p_1 \\ & B & & \vdots & \vdots \\ & & & 1 & p_N \\ -1 & \cdots & -1 & 0 & 0 \\ -p_1 & \cdots & -p_N & 0 & 0 \end{vmatrix}, \tag{A.12}$$

$$-F_\tau + 4aF_{xxx} = 3a\cdot(2i)^{N-1}\begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -p_1 & \cdots & -p_N & 0 \end{vmatrix}, \tag{A.13}$$

$$F_y^2 + F_{xx}^2 = (2i)^{2N-2}\begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -1 & \cdots & -1 & 0 \end{vmatrix}\begin{vmatrix} & & & -1 \\ & B & & \vdots \\ & & & -1 \\ p_1 & \cdots & p_N & 0 \end{vmatrix}, \tag{A.14}$$

$$2abF_{xxxy} - 3aF_{xt} + 3aF_{x\tau} + bF_{y\tau} = \frac{3}{2}ab\cdot(2i)^{N-2}\left(\begin{vmatrix} & & & 1 & p_1 \\ & B & & \vdots & \vdots \\ & & & 1 & p_N \\ p_1^2 & \cdots & p_N^2 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} & & & p_1^2 & -1 \\ & B & & \vdots & \vdots \\ & & & p_N^2 & -1 \\ -1 & \cdots & -1 & 0 & 0 \\ p_1 & \cdots & p_N & 0 & 0 \end{vmatrix}\right), \tag{A.15}$$

$$-6abF_{xxy} + 3aF_t - 3aF_\tau = \frac{3}{2}ab\cdot(2i)^{N-1}\left(\begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ p_1^2 & \cdots & p_N^2 & 0 \end{vmatrix} + \begin{vmatrix} & & & p_1^2 \\ & B & & \vdots \\ & & & p_N^2 \\ p_1 & \cdots & p_N & 0 \end{vmatrix}\right), \tag{A.16}$$

$$-2abF_{xxx} - bF_\tau = \frac{3}{2}ab\cdot(2i)^{N-1}\left(\begin{vmatrix} & & & p_1^2 \\ & B & & \vdots \\ & & & p_N^2 \\ -1 & \cdots & -1 & 0 \end{vmatrix} - \begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ p_1^2 & \cdots & p_N^2 & 0 \end{vmatrix}\right). \tag{A.17}$$

Substituting (A.8)–(A.17) to equations (4) and (5) gives the Jacobi identities for determinants:

$$\begin{aligned}
&\left[D_xD_\tau + a\left(3D_y^2 - D_x^4\right)\right]F\cdot F = 2\left(F_{x\tau}F - F_xF_\tau + 3aF_{yy}F - 3aF_y^2 - 3aF_{xx}^2 + 4aF_xF_{xxx} - aFF_{xxxx}\right) \\
&= 2(2i)^{2N-2}\left(-3a\begin{vmatrix} & & & 1 & p_1 \\ & B & & \vdots & \vdots \\ & & & 1 & p_N \\ -1 & \cdots & -1 & 0 & 0 \\ -p_1 & \cdots & -p_N & 0 & 0 \end{vmatrix}|B| + 3a\begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -p_1 & \cdots & -p_N & 0 \end{vmatrix}\cdot\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix} - 3a\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ -p_1 & \cdots & -p_N & 0 \end{vmatrix}\begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -1 & \cdots & -1 & 0 \end{vmatrix}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\left[a\left(2bD_x^3D_y - 3D_xD_t + 3D_xD_\tau\right) + bDyD_\tau\right]F\cdot F \\
&= 2\left[2ab\left(F_{xxxy}F - 3F_{xxy}F_x - F_{xxx}F_y + 3F_{xx}F_{xy}\right) - 3a(F_{xt}F - F_xF_t) + 3a(F_{x\tau}F - F_xF_\tau) + b\left(F_{y\tau}F - F_yF_\tau\right)\right] \\
&= 3ab(2i)^{2N-2}\left(\begin{vmatrix} & & & 1 & p_1^2 \\ & B & & \vdots & \vdots \\ & & & 1 & p_N^2 \\ p_1^2 & \cdots & p_N^2 & 0 & 0 \\ -1 & \cdots & -1 & 0 & 0 \end{vmatrix}|B| + \begin{vmatrix} & & & p_1^2 & 1 \\ & B & & \vdots & \vdots \\ & & & p_N^2 & 1 \\ -1 & \cdots & -1 & 0 & 0 \\ p_1 & \cdots & p_N & 0 & 0 \end{vmatrix}|B| + \begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ p_1^2 & \cdots & p_N^2 & 0 \end{vmatrix}\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix} + \begin{vmatrix} & & & p_1^2 \\ & B & & \vdots \\ & & & p_N^2 \\ p_1 & \cdots & p_N & 0 \end{vmatrix}\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ -1 & \cdots & -1 & 0 \end{vmatrix} \right. \\
&\left. - \begin{vmatrix} & & & p_1^2 \\ & B & & \vdots \\ & & & p_N^2 \\ -1 & \cdots & -1 & 0 \end{vmatrix}\begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ p_1 & \cdots & p_N & 0 \end{vmatrix} - \begin{vmatrix} & & & 1 \\ & B & & \vdots \\ & & & 1 \\ p_1^2 & \cdots & p_N^2 & 0 \end{vmatrix}\begin{vmatrix} & & & p_1 \\ & B & & \vdots \\ & & & p_N \\ -1 & \cdots & -1 & 0 \end{vmatrix}\right) = 0.
\end{aligned} \tag{A.18}$$

B. Proof of Proposition 2

In this appendix, we give the detailed proof of Proposition 2 through the determinant technique. We can derive the following differential formulae for determinant G by applying the dispersion relations (19) and (20):

$$G_x = \begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix},$$

$$G_{xx} = \begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix} + \begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix},$$

$$G_y = i\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix} - i\begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix},$$

$$G_{xy} = -i\begin{vmatrix} G & \begin{matrix} f_{1,xx} \\ \vdots \\ f_{N,xx} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix} - i\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} g_{1,xx} & \cdots & g_{N,xx} \end{matrix} & 0 \end{vmatrix},$$

$$G_{x\tau} + 3aG_{yy} - aG_{xxxx} = 12a\begin{vmatrix} G & \begin{matrix} f_1 & f_{1,x} \\ \vdots & \vdots \\ f_N & f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \\ g_{1,x} & \cdots & g_{N,x} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{vmatrix},$$

$$-G_\tau + 4aG_{xxx} = 12a\begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix},$$

$$G_y^2 + G_{xx}^2 = 4\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix},$$

$$2abG_{xxxy} - 3aG_{xt} + 3aG_{x\tau} + bG_{y\tau}$$
$$= 12abi\left(\begin{vmatrix} G & \begin{matrix} f_{1,xx} & f_1 \\ \vdots & \vdots \\ f_{N,xx} & f_N \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \\ g_{1,x} & \cdots & g_{N,x} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{vmatrix} + \begin{vmatrix} G & \begin{matrix} f_1 & f_{1,x} \\ \vdots & \vdots \\ f_N & f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \\ -g_{1,xx} & \cdots & -g_{N,xx} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{vmatrix}\right),$$

$$-6abG_{xxy} + 3aG_t - 3aG_\tau$$
$$= 12abi\left(\begin{vmatrix} G & \begin{matrix} f_{1,xx} \\ \vdots \\ f_{N,xx} \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix} + \begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} g_{1,xx} & \cdots & g_{N,xx} \end{matrix} & 0 \end{vmatrix}\right),$$

$$-2abG_{xxx} - bG_\tau$$
$$= -6ab\left(\begin{vmatrix} G & \begin{matrix} f_{1,xx} \\ \vdots \\ f_{N,xx} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix} + \begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_{1,xx} & \cdots & -g_{N,xx} \end{matrix} & 0 \end{vmatrix}\right). \tag{B.1}$$

Substituting equations (A.2)–(A.12) into equations (4) and (5) gives the Jacobi identities for determinants:

$$\left[D_xD_\tau + a\left(3D_y^2 - D_x^4\right)\right]G\cdot G = 2\left(G_{x\tau}G - G_xG_\tau + 3aG_{yy}G - 3aG_y^2 - 3aG_{xx}^2 + 4aG_xG_{xxx} - aGG_{xxxx}\right) = 24a\left(\begin{vmatrix} G & \begin{matrix} f_1 & f_{1,x} \\ \vdots & \vdots \\ f_N & f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \\ g_{1,x} & \cdots & g_{N,x} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{vmatrix}|G| - \begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} g_{1,x} & \cdots & g_{N,x} \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix} + \begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} g_{1,x} & \cdots & g_{N,x} \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix}\right) = 0,$$

$$\left[a\left(2bD_x^3D_y - 3D_xD_t + 3D_xD_\tau\right) + bDyD_\tau\right]G\cdot G = 2\left[2ab\left(G_{xxxy}G - 3G_{xxy}G_x - G_{xxx}G_y + 3G_{xx}G_{xy}\right) - 3a(G_{xt}G - G_xG_t) + 3a(G_{x\tau}G - G_xG_\tau) + b\left(G_{y\tau}G - G_yG_\tau\right)\right]$$
$$= 24abi\left(\begin{vmatrix} G & \begin{matrix} f_{1,xx} & f_1 \\ \vdots & \vdots \\ f_{N,xx} & f_N \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \\ -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{vmatrix}|G| + \begin{vmatrix} G & \begin{matrix} f_1 & 1 \\ \vdots & \vdots \\ f_N & 1 \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \\ -g_{1,xx} & \cdots & -g_{N,xx} \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{vmatrix}|G| + \begin{vmatrix} G & \begin{matrix} f_{1,xx} \\ \vdots \\ f_{N,xx} \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix} - \begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_{1,xx} & \cdots & -g_{N,xx} \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix} - \begin{vmatrix} G & \begin{matrix} f_{1,xx} \\ \vdots \\ f_{N,xx} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_{1,x} & \cdots & -g_{N,x} \end{matrix} & 0 \end{vmatrix} + \begin{vmatrix} G & \begin{matrix} f_1 \\ \vdots \\ f_N \end{matrix} \\ \begin{matrix} -g_{1,xx} & \cdots & -g_{N,xx} \end{matrix} & 0 \end{vmatrix}\begin{vmatrix} G & \begin{matrix} f_{1,x} \\ \vdots \\ f_{N,x} \end{matrix} \\ \begin{matrix} -g_1 & \cdots & -g_N \end{matrix} & 0 \end{vmatrix}\right) = 0. \tag{B.2}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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