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Analysis of Functional Properties of Fuzzy $\widetilde{L}^p(\Omega)$ Spaces

Grace NKWESE MAZONI [a*], Clara PALUKU KASOKI [a], Emilien LORANU LONDJIRINGA [b], Jonathan OPFOINTSHI ENGOMBANGI [c] and Camile LIKOTELO BINENE [a]

[a] *Department of Mathematics and Computer Science, Faculty of Science and Technology, National Pedagogical University, Kinshasa, DRC.*
[b] *Department of Mathematics Physics, Exact Sciences Section, ISP BUNIA, Ituri, DRC.*
[c] *Higher Institute of Medical Techniques of Bandundu (ISTM BDD), Democratic Republic of Congo.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This paper proposes a study of fuzzy $\widetilde{L}^p(\Omega)$ spaces, incorporating functions with triangular fuzzy coefficients. These fuzzy functional spaces provide a better adaptation to fuzzy or imprecise functions. We establish the theoretical foundations of these spaces by examining key functional properties, such as the fuzzy scalar product and the fuzzy norm.

**Corresponding author: Email: nkwesegrace816@gmail.com;*

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To do this, we checked the bilinearity, symmetry, positivity, homogeneity and triangular inequality in a fuzzy environment and in the presence of functions whose coefficients are triangular fuzzy numbers, by the α-cut Dubois and Prade approach.
The aim of this paper is to address observations identified in the existing literature, where some functional properties of $\widetilde{L}^p(\Omega)$ fuzzy spaces are often addressed in a too general manner, without specifying the types of fuzzy functions. This study aims to provide a more detailed and rigorous analysis, thus enriching mathematical understanding and paving the way for practical applications in diverse fields such as: fuzzy differential equations, artificial intelligence, information processing, and decision making in uncertain environments.

Keywords: α-cuts; fuzzy functions; fuzzy integrals; $\widetilde{L}^p(\Omega)$ fuzzy spaces; fuzzy scalar product and fuzzy norm.

1 Introduction

The classical $L^p(\Omega)$ spaces, defined by
$L^p(\Omega) = \left\{f: \Omega \longrightarrow \mathbb{R}, f \text{ integrable and } \int_\Omega |f(x)|^p dx < +\infty\right\}$, are fundamental tools in functional analysis, differential equation theory and probability theory. However, their application to fuzzy functions or fuzzy measures reveals significant limitations. To overcome these limitations, the study of $\widetilde{L}^p(\Omega)$ fuzzy spaces, which uses fuzzy integrals, has been considered.

In the existing literature on fuzzy norms and integrals, several relevant contributions have been observed [1]. In 2013, Chamkha Fatima Zohra examined the scalar product and the norm of fuzzy numbers using the α-cuts approach, but without extending these concepts to fuzzy functions. Thus, her contribution remains limited with respect to the definition of $\widetilde{L}^p(\Omega)$ fuzzy spaces in a context of fuzzy functions [2]. In 2020, Abbas Ghaffari et al. proposed a dynamic model of fuzzy measure theory, called $*$-fuzzy measure, and showed how this model, based on fuzzy sets and triangular norms, allows to define a norm on $*$-fuzzy function spaces $(L^+)^p$ and to prove inequalities such as Chebyshev's and H's ölder. However, this approach does not directly apply to $\widetilde{L}^p(\Omega)$ fuzzy spaces when the functions have triangular fuzzy coefficients, as it is based only on the membership degree.

Faced with these limitations, our study proposes to fill these observations by defining the spaces $\widetilde{L}^p(\Omega)$ in the framework of triangular fuzzy coefficient functions, using the Dubois and Prade α-cut approach. This analysis aims to deepen the theoretical understanding of fuzzy functional spaces and to provide a more complete and rigorous approach to the associated functional properties.

2 Fuzzy Arithmetic of α-cuts$[2, 3, 4, 5]$

2.1 α-cuts, kernel and support of a triangular fuzzy number

Definition 2.1.1. Let be $\widetilde{A} = (a, b, c)$a triangular fuzzy number, such that

$a < b < c$.The α-cuts of $\tilde{A}$ are defined by the following relation:

$$\widetilde{A} = [A_\alpha^-, A_\alpha^+]$$

$$= [(b-a)\alpha + a, (b-c)\alpha + c], \alpha \in [0,1]$$

Definition 2.1.2. We call the kernel of $\widetilde{A}$, denoted $N(\widetilde{A})$, the set:

$$N(\widetilde{A}) = \{x \in X: u_{\widetilde{A}}(x) = 1\}$$

With X as universe of discourse.

Definition 2.1.3. We call the support of $\widetilde{A}$, denoted $supp(\widetilde{A})$, the set:

$$supp(\widetilde{A}) = \{x \in X: O \leq u_{\widetilde{A}}(x) \leq 1\}$$

With $u_{\widetilde{A}}(x)$ the membership degree of x to the subset $\widetilde{A}$.

Remark 2.1.4. In practice, for a triangular fuzzy number

$$\widetilde{A} = (a, b, c):$$

(i) if $\alpha = 0$ $then$ $[A_0^-, A_0^+] = [a, c] = supp(\widetilde{A})$

(ii) if $\alpha = 1$ $then$ $[A_1^-, A_1^+] = \{b\} = N(\widetilde{A})$

Example 2.1.5. Let a triangular fuzzy number be

$$\widetilde{A} = (1,2,3)$$

$\widetilde{A} = [(2-1)\alpha + 1, (2-3)\alpha + 3] = [\alpha + 1, -\alpha + 3], \alpha \in [0,1]$

For $\alpha = 0 \Rightarrow supp(\tilde{A}) = [1,3]$

For $\alpha = 1 \Rightarrow N(\tilde{A}) = \{2\}$.

2.2 Operations on the α-cuts $[2, 3, 4, 6, 5]$

Let $\widetilde{A} = (a_1, c_1, b_1)$ and $\widetilde{B} = (a_2, c_2, b_2)$ be two triangular fuzzy numbers defined by their respective α-cuts: $\widetilde{A} = [a_1, b_1]$ and $\widetilde{B} = [a_2, b_2]$. We can then perform the following operations:

(i)Multiplication

$$\widetilde{A} \otimes \widetilde{B} = [a_1, b_1] \otimes [a_2, b_2] = [Min\ G, Max\ G]$$

Where G is defined by $G = \{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}$.

(ii)Addition

$$\widetilde{A} \oplus \widetilde{B} = [a_1, b_1] \oplus [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$$

(iii)Subtraction

$$\widetilde{A} \ominus \widetilde{B} = [a_1, b_1] \ominus [a_2, b_2] = [a_1 - b_2, b_1 - a_2]$$

(iv)Division

$$\frac{\widetilde{A}}{\widetilde{B}} = \frac{[a_1, b_1]}{[a_2, b_2]} = \left[Min\ \left(\frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2}\right), Max\ \left(\frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2}\right)\right] (\widetilde{B} \neq 0)$$

(v)Multiplication by a scalar

Let $\lambda \in \mathbb{R}$ and $\widetilde{A} = [a_1, b_1]$
If $\lambda > 0, \lambda \otimes [a_1, b_1] = [\lambda a_1, \lambda b_1]$
If $\lambda < 0, \lambda \otimes [a_1, b_1] = [\lambda b_1, \lambda a_1]$

3 Fuzzy Functions and Fuzzy Integration $[7, 8, 9, 10, 11, 12]$

3.1 Functions with triangular fuzzy coefficients

Definition 3.1.1. Let be $\mathbb{R}_{\mathcal{F}}$the set of fuzzy numbers. We call a function of a variable with triangular fuzzy coefficients any function $\widetilde{f}$ defined by:

$$\tilde{f}:\mathbb{R}\longrightarrow\mathbb{R}_{\mathcal{F}}$$

Definition 3.1.2. Let be $\mathbb{R}_{\mathcal{F}}$the set of fuzzy numbers. We call a function of several variables with triangular fuzzy coefficients any function $\tilde{f}$defined by:

$$\tilde{f}:\mathbb{R}^n\longrightarrow\mathbb{R}_{\mathcal{F}}$$

The representation of the α-cuts of a fuzzy function is given by:

$$\tilde{f}_{\alpha}(x)=\left[\left(\tilde{f}_{\alpha}(x)\right)^{-},\left(\tilde{f}_{\alpha}(x)\right)^{+}\right],\alpha\in[0,1]$$

3.2 Fuzzy integration

Definition 3.2.1. Let $\tilde{f}$ a function with triangular fuzzy coefficients and $\Omega\subset\mathbb{R}$ its domain of integration be. The fuzzy integral I of $\tilde{f}$ is given by the following relation:

$$I=\int_{\Omega}\tilde{f}(x)dx$$

By the α-cuts approach ,we have:

$$\int_{\Omega}\tilde{f}(x)dx=\left[\int_{\Omega}\left(\tilde{f}_{\alpha}(x)\right)^{-}dx,\int_{\Omega}\left(\tilde{f}_{\alpha}(x)\right)^{+}dx\right],\alpha\in[0,1]$$

3.2.2 Properties

(i) if $\tilde{f},\tilde{g}:[a,b]\longrightarrow\mathbb{R}_{\mathcal{F}}$ are two integrable fuzzy functions and $\ell,k\in\mathbb{R}$, we have:

$$\int_a^b[\ell\tilde{f}(x)+k\tilde{g}(x)]dx=\ell\int_a^b\tilde{f}(x)dx+k\int_a^b\tilde{g}(x)dx.$$

(ii) if $\tilde{f}:[a,b]\rightarrow\mathbb{R}_{\mathcal{F}}$is integrable and $c\in[a,b]$,we have:

$$\int_a^b\tilde{f}(x)dx=\int_a^c\tilde{f}(x)dx+\int_c^b\tilde{f}(x)dx.$$

Example. We give the fuzzy function $\tilde{f}(x)=(1,2,3)\,x$ et $\Omega=[2,5]$. Let us calculate the integral on Ω by the $\tilde{f}(x)$ α-cut .approach

We have:

$$I=\int_2^5(1,2,3)x\,dx.$$

$$Supp\,(I)=\int_2^5[1,3]x\,\,dx$$

$$=\left[\int_2^5x\,dx,\int_2^5 3x\,dx\right]$$

$$=\left[\frac{21}{2},\frac{63}{2}\right]$$

$$N(I) = \int_2^5 2x\, dx = 21.$$

Hence the value of the fuzzy integral I is $\left(\frac{21}{2}, 21, \frac{63}{2}\right)$ or $(10{,}5\ ; 21\ ;\ 31{,}5)$, a triangular fuzzy number.

4 Spaces$\boldsymbol{L^p(\Omega)}$

Definition 4.1. Let $p\ be$ a real number such that $1 \leq p < +\infty$ and Ω an open set in $\mathbb{R}^n$. the $L^p(\Omega)$ space is defined as follows**:**

$$L^p(\Omega) = \left\{f: \Omega \to \mathbb{R}, f\ is \text{ intégrable and } \int_{\boldsymbol{\Omega}} |f(x)|^p dx < +\infty\right\}$$

The norm associated with this space is given by: $\|f\|_{L^p(\Omega)} = \left(\int_{\boldsymbol{\Omega}} |f(x)|^p dx\right)^{\frac{1}{p}}$

Proposition 4.2. The space $\left(L^p(\Omega), \|.\|_{L^p(\Omega)}\right)$ is a normed space for

$1 \leq p < +\infty.$

Definition 4.3. Let be Ω an open set of $\mathbb{R}^n$. The $L^2(\Omega)$ space is defined as follows:

$$L^2(\Omega) = \left\{f: \Omega \to \mathbb{R}, f\ is \text{ integrable and } \int_{\Omega} |f(x)|^2 dx < +\infty\right\}$$

We denote the scalar product in this space by:$\langle f, g\rangle_{L^2(\Omega)} = \int_{\boldsymbol{\Omega}} f(x)g(x)dx, \forall (f, g) \in [L^2(\Omega)]^2$.

The standard associated with this space is:$\|f\|_{L^2(\Omega)} = \left(\int_{\Omega} |f(x)|^2 dx\right)^{\frac{1}{2}}$

Proposition 4.4. $\langle .,. \rangle_{L^2(\Omega)}$ is a scalar product on $L^2(\Omega)$**.**

5 $\boldsymbol{\widetilde{L}^p(\Omega)}$ fuzzy spaces $[\mathbf{7, 8, 9, 10}]$

Definition 5.1. Let Ωan open set of $\mathbb{R}^n$ and p a real such that $1 \leq p < +\infty$.We call the $\tilde{L}^p(\Omega)$ fuzzy space, the set of functions with triangular fuzzy coefficients $\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}}$ such that $\tilde{f}$ is integrable on Ω and we set:

$$\widetilde{L}^p(\Omega) = \left\{\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}}, \tilde{f}\ is \text{ integrable and } \int_{\Omega} \left|\tilde{f}(x)\right|^p dx < +\infty\right\}$$

We recall that in spaces $\widetilde{L}^p(\Omega)$, the scalar product is defined for $p = 2$.

Definition 5.2. Fuzzy scalar products on $\widetilde{L}^2(\Omega)$

Let $\tilde{f}$ and be $\tilde{g}$ two functions with triangular fuzzy coefficients in $\widetilde{L}^2(\Omega)$ and

$\Omega \subset \mathbb{R}$. We define the fuzzy scalar product of $\tilde{f}$ and $\tilde{g}$ by the relation:

$$\langle \tilde{f}, \tilde{g}\rangle_{\widetilde{L}^2(\Omega)} = \int_{\Omega} \tilde{f}(x)\tilde{g}(x)dx.$$

The approach of the α -cuts of $\langle \tilde{f}, \tilde{g}\rangle_{\widetilde{L}^2(\Omega)}$ is given by:

$$\langle \tilde{f}, \tilde{g} \rangle_{\tilde{L}^2(\Omega)} = \left[\int_\Omega [\tilde{f}_\alpha(x)\tilde{g}_\alpha(x)]^- dx , \int_\Omega [\tilde{f}_\alpha(x)\tilde{g}_\alpha(x)]^+ dx \right], \alpha \in [0,1]$$

Definition 5.3. Fuzzy norms in $\tilde{L}^p(\Omega)$

Let $\Omega \subset \mathbb{R}$, $1 \leq p < +\infty$ and be $\tilde{f}$ a fuzzy function with triangular fuzzy coefficients. We have:

$$(i) \|\tilde{f}\|_{\tilde{L}^2(\Omega)} = \sqrt{\langle \tilde{f}, \tilde{f} \rangle_{\tilde{L}^2(\Omega)}} = \left(\int_\Omega [\tilde{f}(x)]^2 dx \right)^{1/2}$$

The α-cuts approach is given by:

$$\|\tilde{f}\|_{\tilde{L}^2(\Omega)} = \left(\left[\int_\Omega (\tilde{f}_\alpha^-(x))^2 dx, \int_\Omega (\tilde{f}_\alpha^+(x))^2 dx \right] \right)^{1/2}, \alpha \in [0,1]$$

$$(ii) \|\tilde{f}\|_{\tilde{L}^p(\Omega)} = \left(\int_\Omega |\tilde{f}(x)|^p dx \right)^{1/p}$$

The α-cuts approach is given by:

$$\|\tilde{f}\|_{\tilde{L}^p(\Omega)} = \left(\left[\int_\Omega |\tilde{f}_\alpha^-(x)|^p dx , \int_\Omega |\tilde{f}_\alpha^+(x)|^p dx \right] \right)^{1/p}, \alpha \in [0,1]$$

Proposition 5.4. The application $\langle .,. \rangle_{\tilde{L}^2(\Omega)}$ is a fuzzy scalar product on $\tilde{L}^2(\Omega)$.

Proof: To show that $\langle .,. \rangle_{\tilde{L}^2(\Omega)}$is a fuzzy scalar product on the space by the $\tilde{L}^2(\Omega)$ α-cut ,approach αwe need to check the properties of bilinearity, symmetry and definite positivity.

$(\boldsymbol{i})$**Bilinearity**

For all $\tilde{f}, \tilde{g}, \tilde{h}, \tilde{p} \in \tilde{L}^2(\Omega)$ and for all $a, b, c, d \in \mathbb{R}$, we have:

$$\langle a\tilde{f} + b\tilde{g}, c\tilde{h} + d\tilde{p} \rangle_{\tilde{L}^2(\Omega)} = ac.\langle \tilde{f}, \tilde{h} \rangle_{\tilde{L}^2(\Omega)} + ad\langle \tilde{f}, \tilde{p} \rangle_{\tilde{L}^2(\Omega)}$$
$$+bc.\langle \tilde{g}, \tilde{h} \rangle_{\tilde{L}^2(\Omega)} + bd\langle \tilde{g}, \tilde{p} \rangle_{\tilde{L}^2(\Omega)}$$
$$\langle a\tilde{f} + b\tilde{g}, c\tilde{h} + d\tilde{p} \rangle_{\tilde{L}^2(\Omega)}$$
$$= \left[\int_\Omega \left[\left(a\tilde{f}_\alpha(x) + b\tilde{g}_\alpha(x) \right) \left(c\tilde{h}_\alpha(x) + d\tilde{p}_\alpha(x) \right) \right]^- dx , \int_\Omega \left[\left(a\tilde{f}_\alpha(x) \right. \right. \right.$$
$$\left. \left. \left. + b\tilde{g}_\alpha(x) \right) \left(c\tilde{h}_\alpha(x) + d\tilde{p}_\alpha(x) \right) \right]^+ dx \right], \alpha \in [0,1]$$

Let's develop the products:

$$\left(a\tilde{f}_\alpha(x) + b\tilde{g}_\alpha(x) \right) \left(c\tilde{h}_\alpha(x) + d\tilde{p}_\alpha(x) \right) = ac\tilde{f}_\alpha(x)\tilde{h}_\alpha(x) +$$

$$ad\,\tilde{f}_\alpha(x)\,\tilde{p}_\alpha(x) + bc\tilde{g}_\alpha(x)\tilde{h}_\alpha(x) + bd\,\tilde{g}_\alpha(x)\tilde{p}_\alpha(x).$$

So, we have:

$$\langle a\tilde{f} + b\tilde{g}, c\tilde{h} + d\tilde{p} \rangle_{\tilde{L}^2(\Omega)} =$$

$$\left[\int_\Omega \left[ac\,\widetilde{f}_\alpha(x)\widetilde{h}_\alpha(x) + ad\widetilde{f}_\alpha(x)\widetilde{p}_\alpha(x) + bc\,\widetilde{g}_\alpha(x)\widetilde{h}_\alpha(x) + bd\widetilde{g}_\alpha(x)\widetilde{p}_\alpha(x)\right]^- dx\,, \int_\Omega \left[ac\,\widetilde{f}_\alpha(x)\widetilde{h}_\alpha(x) + ad\widetilde{f}_\alpha(x)\widetilde{p}_\alpha(x) + bc\,\widetilde{g}_\alpha(x)\widetilde{h}_\alpha(x) + bd\widetilde{g}_\alpha(x)\widetilde{p}_\alpha(x)\right]^+ dx\right]$$

Using the linearity of fuzzy integrals, we obtain:

$$= ac\left[\int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{h}_\alpha(x)\right]^-, \left[\widetilde{f}_\alpha(x)\widetilde{h}_\alpha(x)\right]^+ dx\right]$$

$$+ad\left[\int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{p}_\alpha(x)\right]^- dx, \int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{p}_\alpha(x)\right]^+ dx\right]$$

$$+bc\left[\int_\Omega \left[\widetilde{g}_\alpha(x)\widetilde{h}_\alpha(x)\right]^- dx, \int_\Omega \left[\widetilde{g}_\alpha(x)\widetilde{h}_\alpha(x)\right]^+ dx\right]$$

$$+bd\left[\int_\Omega [\widetilde{g}_\alpha(x)\widetilde{p}_\alpha(x)]^- dx, \int_\Omega [\widetilde{g}_\alpha(x)\widetilde{p}_\alpha(x)]^+ dx\right]$$

$$= ac.\langle \widetilde{f}, \widetilde{h}\rangle_{\widetilde{L}^2(\Omega)} + ad\langle \widetilde{f}, \widetilde{p}\rangle_{\widetilde{L}^2(\Omega)} + bc\langle \widetilde{g}, \widetilde{h}\rangle_{\widetilde{L}^2(\Omega)} + bd\langle \widetilde{g}, \widetilde{p}\rangle_{\widetilde{L}^2(\Omega)}$$

This proves the bilinearity of $\langle .,. \rangle_{\widetilde{L}^2(\Omega)}$.

(ii)Symmetry

For everything $\widetilde{f}, \widetilde{g} \in \widetilde{L}^2(\Omega)$, we must verify that:

$$\langle \widetilde{f}, \widetilde{g}\rangle_{\widetilde{L}^2(\Omega)} = \langle \widetilde{g}, \widetilde{f}\rangle_{\widetilde{L}^2(\Omega)}$$

Let's calculate each side:

$$\langle \widetilde{f}, \widetilde{g}\rangle_{\widetilde{L}^2(\Omega)} = \left[\int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{g}_\alpha(x)\right]^- dx, \int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{g}_\alpha(x)\right]^+ dx\right], \alpha \in [0,1]$$

$$\langle \widetilde{g}, \widetilde{f}\rangle_{\widetilde{L}^2(\Omega)} = \left[\int_\Omega \left[\widetilde{g}_\alpha(x)\widetilde{f}_\alpha(x)\right]^- dx, \int_\Omega \left[\widetilde{g}_\alpha(x)\widetilde{f}_\alpha(x)\right]^+ dx\right], \alpha \in [0,1].$$

As $\widetilde{f}_\alpha(x)\widetilde{g}_\alpha(x) = \widetilde{g}_\alpha(x)\widetilde{f}_\alpha(x)$, the negative and positive parts of the products are the same. Thus:

$$\int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{g}_\alpha(x)\right]^- dx = \int_\Omega \left[\widetilde{g}_\alpha(x)\widetilde{f}_\alpha(x)\right]^- dx\,, \alpha \in [0,1]$$
$$\int_\Omega \left[\widetilde{f}_\alpha(x)\widetilde{g}_\alpha(x)\right]^+ dx = \int_\Omega \left[\widetilde{g}_\alpha(x)\widetilde{f}_\alpha(x)\right]^+ dx\,, \alpha \in [0,1]$$

So, $\langle \widetilde{f}, \widetilde{g}\rangle_{\widetilde{L}^2(\Omega)} = \langle \widetilde{g}, \widetilde{f}\rangle_{\widetilde{L}^2(\Omega)}$this shows symmetry.

(iii)Positivity Defined

For it $\langle \widetilde{f}, \widetilde{f}\rangle_{\widetilde{L}^2(\Omega)}$to be positive, we must show that:

$$\langle \tilde{f}, \tilde{f} \rangle_{\tilde{L}^2(\Omega)} = \left[\int_\Omega [\tilde{f}_\alpha(x)\tilde{f}_\alpha(x)]^- dx, \int_\Omega [\tilde{f}_\alpha(x)\tilde{f}_\alpha(x)]^+ dx \right], \alpha \in [0,1]$$

Since $\tilde{f}_\alpha(x).\tilde{f}_\alpha(x) = \left(\tilde{f}_\alpha(x)\right)^2$, we have:

$$\left[\int_\Omega \left[\left(\tilde{f}_\alpha(x)\right)^2 \right]^- dx, \int_\Omega [(\tilde{f}_\alpha(x)^2)]^+ dx \right], \alpha \in [0,1]$$

The terms $[(\tilde{f}_\alpha(x)^2)]^\pm$ represent the negative and positive parts of $\left(\tilde{f}_\alpha(x)\right)^2$. Since $\left(\tilde{f}_\alpha(x)\right)^2$is positive, its negative part is zero:

$$\left[\left(\tilde{f}_\alpha(x)\right)^2 \right]^- = 0, \text{so} \int_\Omega \left[\left(\tilde{f}_\alpha(x)\right)^2 \right]^- dx = 0, \alpha \in [0,1]$$

The positive part is simply:

$$\int_\Omega \left[\left(\tilde{f}_\alpha(x)\right)^2 \right]^+ dx = \int_\Omega \left(\tilde{f}_\alpha(x)\right)^2 dx, \alpha \in [0,1]$$

$$\text{As} \int_\Omega \left(\tilde{f}_\alpha(x)\right)^2 dx \geq 0, \text{the fuzzy inner product is :}$$

$$\langle \tilde{f}, \tilde{f} \rangle_{\tilde{L}^2(\Omega)} = \left[0, \int_\Omega \left(\tilde{f}_\alpha(x)\right)^2 dx \right], \alpha \in [0,1]$$

Moreover, $\langle \tilde{f}, \tilde{f} \rangle_{\tilde{L}^2(\Omega)} = [0,0]$if and only if $\tilde{f} = 0$ on Ω because$\int_\Omega \left(\tilde{f}_\alpha(x)\right)^2 dx = 0$ implies $\tilde{f}_\alpha = 0$ almost everywhere on Ω.Therefore, the definite positivity is well verified.

Under (i), (ii) et (iii), $\langle .,. \rangle_{\tilde{L}^2(\Omega)}$is a fuzzy dot product on$\widetilde{L}^2(\Omega)$. ■

Proposition 5.5. Space $\left(\tilde{L}^p(\Omega), \|.\|_{\tilde{L}^p(\Omega)}\right)$ is a fuzzy normed space for

$p \geq 1.$

Proof: To show that $\left(\tilde{L}^p(\Omega), \|.\|_{\tilde{L}^p(\Omega)}\right)$is a fuzzy normed space, we must prove that $\|.\|_{\tilde{L}^p(\Omega)}$ satisfies the three properties of a norm on $\tilde{L}^p(\Omega)$

The fuzzy norm $\|.\|_{\tilde{L}^p(\Omega)}$ is defined by the α-cuts approach as follows :

$$\|\tilde{f}\|_{\tilde{L}^p(\Omega)} = \left([\int_\Omega |\tilde{f}_\alpha^-(x)|^p dx, \int_\Omega |\tilde{f}_\alpha^+(x)|^p dx]\right)^{1/p}, \alpha \in [0,1]$$

For everything $\tilde{f} \in \tilde{L}^p(\Omega)$..

(i)Positivity

$$\|\tilde{f}\|_{\tilde{L}^p(\Omega)} \geq 0 \text{ pour tout } \tilde{f} \in \tilde{L}^p(\Omega), \text{et } \|\tilde{f}\|_{\tilde{L}^p(\Omega)} = 0 \text{ if and only if } \tilde{f} = 0.$$

For everything $\tilde{f} \in \tilde{L}^p(\Omega)$, we have:

$$\|\tilde{f}\|_{\tilde{L}^p(\Omega)} = \left(\left[\int_\Omega |[\tilde{f}_\alpha(x)]^-|^p dx, \int_\Omega \left|[\tilde{f}_\alpha(x)]^+\right|^p dx \right] \right)^{1/p}, \alpha \in [0,1]$$

The integrals $\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx \geq 0$ et $\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx \geq 0$.

Therefore, $\|\widetilde{f}\|_{\widetilde{L}^p(\Omega)} > 0$.

To $\|\widetilde{f}\|_{\widetilde{L}^p(\Omega)} = 0$ imply that $\widetilde{f} = 0$, with $\|\widetilde{f}\|_{\widetilde{L}^p(\Omega)} = [0,0]$

This means that:

$$\left[\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx, \int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx\right] = [0,0], \alpha \in [0,1]$$

The positivity is therefore verified.

(ii) Homogeneity

For $\lambda \in \mathbb{R}$ and a fuzzy function $\widetilde{f} \in \widetilde{L}^p(\Omega)$, we must demonstrate that:

$$\|\lambda\widetilde{f}\|_{\widetilde{L}^p(\Omega)} = |\lambda|\|\widetilde{f}\|_{\widetilde{L}^p(\Omega)}$$

$$\|\lambda\widetilde{f}\|_{\widetilde{L}^p(\Omega)} = \left(\left[\int_\Omega \left|\lambda\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx, \int_\Omega \left|\lambda\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx\right]\right)^{1/p}, \alpha \in [0,1]$$

$$= \left(\left[\int_\Omega |\lambda|^p\left|\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx, \int_\Omega |\lambda|^p\left|\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx\right]\right)^{1/p}$$

$$= \left(|\lambda|^p\left[\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx, \int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx\right]\right)^{1/p}$$

$$= |\lambda| \left(\left[\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx, \int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx\right]\right)^{1/p}$$

$$= |\lambda| . \|\widetilde{f}\|_{\widetilde{L}^p(\Omega)}$$

Homogeneity is therefore verified.

(iii) Triangular inequality

For all fuzzy functions $\widetilde{f}$ and $\widetilde{g}$ in $\widetilde{L}^p(\Omega)$, we must show that:

$$\|\widetilde{f} + \widetilde{g}\|_{\widetilde{L}^p(\Omega)} \leq \|\widetilde{f}\|_{\widetilde{L}^p(\Omega)} + \|\widetilde{g}\|_{\widetilde{L}^p(\Omega)}$$

Let's calculate $\|\widetilde{f} + \widetilde{g}\|_{\widetilde{L}^p(\Omega)}$:

$$\|\widetilde{f} + \widetilde{g}\|_{\widetilde{L}^p(\Omega)} = \left(\left[\int_\Omega \left|\left[\widetilde{f}_\alpha(x) + \widetilde{g}_\alpha(x)\right]^-\right|^p dx, \int_\Omega \left|\left[\widetilde{f}_\alpha(x) + \widetilde{g}_\alpha(x)\right]^+\right|^p dx\right]\right)^{1/p} \alpha \in [0,1]$$

Let us use Minkowski's inequality for integrals valid for $p \geq 1$:

$$\left(\int_\Omega \left|\left[\widetilde{f}_\alpha(x) + \widetilde{g}_\alpha(x)\right]^-\right|^p dx\right)^{1/p} \leq \left(\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^-\right|^p dx\right)^{1/p} + \left(\int_\Omega |[\widetilde{g}_\alpha(x)]^-|^p dx\right)^{1/p}$$

$$\left(\int_\Omega \left|\left[\widetilde{f}_\alpha(x) + \widetilde{g}_\alpha(x)\right]^+\right|^p dx\right)^{1/p} \leq \left(\int_\Omega \left|\left[\widetilde{f}_\alpha(x)\right]^+\right|^p dx\right)^{1/p} + \left(\int_\Omega |[\widetilde{g}_\alpha(x)]^+|^p dx\right)^{1/p}$$

Combining these two results:

$$\|\tilde{f} + \tilde{g}\|_{\tilde{L}^p(\Omega)} = \left(\left[\int_\Omega \left|[\tilde{f}_\alpha(x) + \tilde{g}_\alpha(x)]^-\right|^p dx, \int_\Omega \left|[\tilde{f}_\alpha(x) + \tilde{g}_\alpha(x)]^+\right|^p dx\right]\right)^{1/p}$$

$$\leq \left(\left[\int_\Omega \left|[\tilde{f}_\alpha(x)]^-\right|^p dx, \int_\Omega \left|[\tilde{f}_\alpha(x)]^+\right|^p dx\right]\right)^{1/p}$$

$$+ \left(\left[\int_\Omega |[\tilde{g}_\alpha(x)]^-|^p dx, \int_\Omega |[\tilde{g}_\alpha(x)]^+|^p dx\right]\right)^{1/p}$$

$$\leq \|\tilde{f}\|_{\tilde{L}^p(\Omega)} + \|\tilde{g}\|_{\tilde{L}^p(\Omega)}.$$

The triangular integrality is therefore verified.

All properties of a fuzzy norm are satisfied, $\left(\tilde{L}^p(\Omega), \|.\|_{\tilde{L}^p(\Omega)}\right)$ is a fuzzy normed space. ■

6 Conclusion

This paper has provided an in-depth analysis of the functional properties of fuzzy spaces $\tilde{L}^p(\Omega)$, incorporating triangular fuzzy coefficient functions. We have established the theoretical foundations of these spaces by examining key functional properties such as fuzzy inner products and fuzzy norms. Using the approach of α-cuts by Dubois and Prade, we demonstrated bilinearity, symmetry, positivity, homogeneity, and the triangular inequality within a fuzzy context, addressing gaps identified in the existing literature. Our results indicate that fuzzy spaces $\tilde{L}^p(\Omega)$offer a more flexible and suitable framework for dealing with fuzzy or imprecise functions, paving the way for various practical applications, including fuzzy differential equations, artificial intelligence, information processing, and decision-making in uncertain environments. To further explore these results, several research directions can be considered:

- Extensions of fuzzy spaces $\tilde{L}^p(\Omega)$
- Study of fuzzy spaces $\tilde{L}^p(\Omega)$ and fuzzy differential equations
- Analysis of advanced functional properties of fuzzy spaces $\tilde{L}^p(\Omega)$
- Development of algorithms based on fuzzy spaces $\tilde{L}^p(\Omega)$for tasks in artificial intelligence

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Chamkha Fatima Zohra: On cauchy problems for fuzzy differential equations. Master's degree final project, Abou Bekr Belkaid–Tlemcen- University , Algeria; 2013.

[2] Abbas Ghaffari et al. Inequalities in triangular norm-based ∗-Fuzzy $(L^+)^p$ Spaces", Article; November 6, 2020.

[3] wang K, Lee H. Fisrt course on fuzzy theory and applications, Springer, Berlin; 2005.

[4] Dubios D, H. Prade H, operations on fuzzy Numbers. International Journal of Systems Science. 1978;9(6):613-626.

[5] Barnabàs Bede. Mathematics of fuzzy sets and fuzzy logic, studies in fuzziness and soft computing, Springer; 2013.

[6] Bede B, Mathematics of fuzzy sets fuzzy logic, Springer-Verlag NY; 2013.

[7] Vinoliah EM, Ganesan K, Fuzzy optimal solution for a fuzzy assignment problem with actagonal fuzzy numbers, IOP Conf. Series. Journal of Physis: Conf. Series. 1000;2018.

[8] Guang-Quan Z, Fuzzy continuous functions and its properties, Fuzzy Sets and Systems. 1991;43(2):159-171.

[9] Allahviranloo T, Gouyandeh Z, Armand A, Hasanoglu A. on fuzzy solution for heat equation based on generalized hukuhara differentiability, Fuzzy Sets and Systems. 2015;265:1-23.

[10] Bede B, Stefanini L. Generalized differentiability of fuzzy-valued functions, Fuzzy sets and systems. 2013;230:119-141.

[11] Wu H-C, The fuzzy riemann integral and its numerical integration, Fuzzy Sets and Systems. 2000;110:1-25.

[12] H. Chung Wu, Fuzzy valued integrals based on a constructive methodology,Applications of Mathematics. 2007;52:1-23.

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