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Numerical Solution of First Order Fuzzy Differential Equations by Simpson's Rule

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Authors' contributions

This work was carried out in collaboration between the authors SSD and KG. Author KG designed the study, wrote the protocol and supervised the work. Author SSD carried out all laboratories work and performed the statistical analysis. Author KG managed the analyses of the study. Author SSD wrote the first draft of the manuscript. Author KG managed the literature searches and edited the manuscript. Both authors read and approved the final manuscript.

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Review Article

Abstract

The concept of fuzzy versions of Simpson's rule and Runge-Kutta method of order four are introduced. In this paper, the solution of fuzzy ordinary differential equation of the first order by Simpson's rule and Runge-Kutta method of order four is presented without converting them to crisp form. The results from these two methods are proved identical by complete error analysis. The accuracy and efficiency of the proposed methods are illustrated by an example with a trapezoidal fuzzy number and triangular fuzzy number.

Keywords: Fuzzy number; trapezoidal fuzzy number; fuzzy differential equations; Runge–Kutta method; higher order derivatives; Simpson's rule.

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1 Introduction

Fuzzy differential equations (FDEs) have wide range of applications in many branches of engineering and in the field of medicine. The concept of fuzzy derivative was first introduced by Chang and Zadeh [1]. Later, Dubois and Prade [2] presented the concept of fuzzy derivative based on the extension principle. Kandel and Byatt [3] introduced the concept of fuzzy differential equation in 1987. The FDEs and the initial value problem were regularly treated by Kaleva [4],[5]. There are several approaches for solving fuzzy differential equations are proposed in the literature. The first and foremost popular one is Hukuhara derivative made by Puri and Ralesu [6]. Here the solution of fuzzy differential equation becomes fuzzier as time goes on. This approach does not reproduce the rich and varied behavior of ordinary differential equations. Bede [7],[8] introduced a strongly generalized differentiability of fuzzy functions. Under this interpretation, the solution of a fuzzy differential equation becomes less fuzzier as time goes on. Ming Ma et al. [9] proposed Euler's method for the numerical solution of fuzzy differential equations. Abbasbandy and Allviranloo [10],[11] proposed Taylor's method and the fourth order Runge-Kutta method for the numerical solution of fuzzy differential equations.

Parimala et al. [12] have proposed second order Runge-Kutta method to solve fuzzy differential equations with fuzzy initial conditions. Palligkinis et al. [13] applied the Runge-Kutta method for more general problems and proved the convergence for n-stage Runge-Kutta method. Nieto et al. [14] showed that any suitable numerical method for ordinary differential equations can be applied to solve numerically fuzzy differential equations under generalized differentiability, and also they implemented the generalized Euler approximation method for solving first order linear fuzzy differential equations. Jayakumar et al. [20][15] studied numerical solutions of fuzzy differential equations by Runge -Kutta method of order five. Kanagarajan et al. [20][16] studied numerical solution of fuzzy differential equations by Milne's predictor-corrector method and the dependency problem. Recently, Ghazanfari et al. [17] have considered Seikkala's derivative and applied a numerical algorithm for solving first order fuzzy differential equation, based on extended Runge-Kutta-like formulae of order 4. The dependency problem in fuzzy computation was discussed by Ahmad and Hasan [18][20] and they used Euler's method based on Zadeh's extension principle for finding the numerical solution of fuzzy differential equations. Kanagarajan and Suresh [20] studied fuzzy differential equations using the concept of generalized differentiability applying improved Euler's method and present the generalized characterization theorem. In this paper, based on the H-difference, we propose the fuzzy versions of fourth order Runge–Kutta method and Simpson's rule for the solution of the first order fuzzy ordinary differential equations without converting them to crisp form. We provide some examples and compare the results with exact solution followed by complete error analysis.

2 Preliminaries

In this section, we recall some basic definition and concepts which will be highly useful throughout this paper.

Definition 2.1. A fuzzy set $\tilde{a}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function $\tilde{a}: R \to [0,1]$ has the following:

(i). $\tilde{a}$ is convex, i.e. $\tilde{a}\{\lambda x_1+(1-\lambda)x_2\} \geq \min\{\tilde{a}(x_1), \tilde{a}(x_2)\}$, for all $x_1, x_2 \in R$ and $\lambda \in [0,1]$.

(ii). $\tilde{a}$ is normal i.e. there exists an $x \in R$ such that $\tilde{a}(x) = 1$.

(iii). $\tilde{a}$ is piecewise continuous. Studied.

Definition 2.2. A trapezoidal fuzzy number is denoted as $\tilde{a} = (a_1, a_2, a_3, a_4)$ and is defined by the membership function.

$$\tilde{a}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}, & a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4. A Trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$ is said to be zero trapezoidal fuzzy number if $a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0.$

Definition 2.5. A Trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$ is said to be non-negative trapezoidal fuzzy number if $a_1 > 0.$

Definition 2.6. Two Trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ are said to be equal trapezoidal fuzzy numbers if $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4$.

2.1 Arithmetic operations on fuzzy numbers

A trapezoidal fuzzy number $\tilde{a} \in F(R)$ can also be represented as a pair $\tilde{a} = \left(\underline{a}, \overline{a}\right)$ of functions $\underline{a}(r)$ and $\overline{a}(r)$ for $0 \le r \le 1$ which satisfies the following requirements:

(i). $\underline{a}(r)$ is a bounded monotonic increasing right continuous function.
(ii). $\overline{a}(r)$ is a bounded monotonic decreasing left continuous function.
(iii). $\underline{a}(r) \le \overline{a}(r),\ 0 \le r \le 1.$

Bede and Gal proposed a new fuzzy arithmetic based upon both location index and fuzziness index functions. The location index number is taken in the ordinary arithmetic, whereas the fuzziness index functions are considered as follows. For arbitrary $\tilde{a} = \left(\underline{a}, \overline{a}\right)$, $\tilde{b} = \left(\underline{b}, \overline{b}\right)$ and k>0 we define addition $\tilde{a} + \tilde{b}$, subtraction $\tilde{a} - \tilde{b}$, and scalar multiplication by $k\tilde{a}$ as follows.

1) $\tilde{a} + \tilde{b} = (\underline{a} + \underline{b}, \overline{a} + \overline{b})$
2) $\tilde{a} - \tilde{b} = (\underline{a} - \overline{b}, \overline{a} - \underline{b})$
3) $k\tilde{a} = (k\underline{a}, k\overline{a})$ for $k \ge 0$
4) $k\tilde{a} = (k\overline{a}, k\underline{a})$ for $k < 0$

According to Zedeh's extension principle [22], If $u, v \in F(R^n)$ and $\lambda \in R$ then $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda [u]^\alpha\ \forall \alpha \in [0,1]$.

In this paper, we use an arbitrary fuzzy number with compact support by a pair of functions $(\underline{a}(r), \overline{a}(r)), (\underline{b}(r), \overline{b}(r))\ \ 0 \le r \le 1$. Let E be the space of fuzzy numbers.

Hausdorff distance between two fuzzy numbers is a mapping $D_H : E \times E \to R_+$ defined by $D_H(a,b) = \sup_{\alpha \in [0,1]} \max\left\{ |\underline{a} - \underline{b}|, |\overline{a} - \overline{b}| \right\}$. It is easy to see that D_H is a metric in E and has the following properties .

(i) $D_H(a+c, b+c) = D_H(a,b), \forall a,b,c \in E,$

(ii) $D_H(ka, kb) = |k| D_H(a,b), \forall k \in R, a,b \in E,$

(iii) $D_H(a+b, c+d) \le D_H(a,c) + D_H(b,d), \forall a,b,c,d \in E,$ *and (D_H, E) is a complete metric space.*

Definition 2.7. Let be $u, v \in R$. If there exists $w \in R$ such that $u = v \oplus w$, then w is called the H-Difference of u and v and is denoted by $u \Theta v$.

Definition 2.8. (Hukuhara Derivative) [23].

Consider a fuzzy mapping $F : (a,b) \to R$ and $t_0 \in (a,b)$. We say that F is differentiable at $t_0 \in (a,b)$ if there exists an element $F'(t_0) \in R$ such that for all $h > 0$ sufficiently small $\exists F(t_0 + h) \Theta F(t_0), F(t_0) \Theta F(t_0 - h)$ and the limits (in the metric D).

$$\lim_{h \to 0+} \frac{F(t_0 + h) \Theta F(t_0)}{h} = \lim_{h \to 0-} \frac{F(t_0) \Theta F(t_0 - h)}{h}$$ exist and are equal to $F'(t_0)$.

Note that this definition of the derivative is very restrictive; for instance in [7] the authors showed that if $F(t) = c.g(t)$ where c a fuzzy number is and $g : [a,b] \to R^+$ is a function with $g'(t) < 0$, then F is not differentiable. To avoid this difficulty, the authors of [7] introduced a more general definition of the derivative for fuzzy mappings.

Definition 2.9. (Generalized Fuzzy Derivative)

Let $F : (a,b) \to R$ and $t_0 \in (a,b)$. we say that F is strongly generalized differentiable at t_0 if there exists as element $F'(t_0) \in R$ such that,

(i). For $h > 0$ sufficiently small $\exists F(t_0 + h) \Theta F(t_0), F(t_0) \Theta F(t_0 - h)$, and the limits satisfy
$$\lim_{h \to 0} \frac{F(t_0 + h) \Theta f(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0) \Theta F(t_0 - h)}{h} = F'(t_0)$$

(ii). For $h > 0$ sufficiently small $\exists F(t_0) \Theta F(t_0 + h), F(t_0 - h) \Theta F(t_0)$, and the limits satisfy
$$\lim_{h \to 0} \frac{F(t_0) \Theta F(t_0 + h)}{(-h)} = \lim_{h \to 0} \frac{F(t_0 - h) \Theta F(t_0)}{(-h)} = F'(t_0)$$

(iii). For $h > 0$ sufficiently small $\exists F(t_0 + h) \Theta F(t_0), F(t_0 - h) \Theta F(t_0)$ and the limits satisfy
$$\lim_{h \to 0} \frac{F(t_0 + h) \Theta f(t_0)}{h} = \lim_{h \to 0} \frac{F(t_0 - h) \Theta F(t_0)}{(-h)} = F'(t_0)$$

(iv). For $h>0$ sufficiently small $\exists F(t_0+h)\,\Theta\, F(t_0), F(t_0-h)\,\Theta\, F(t_0)$ and the limits satisfy $\lim_{h\to 0}\frac{F(t_0)\,\Theta\, f(t_0+h)}{(-h)} = \lim_{h\to 0}\frac{F(t_0)\,\Theta\, F(t_0-h)}{h} = F'(t_0)$ h and (-*h*) at denominators mean $\frac{1}{h}$ and - $\frac{1}{h}$ respectively.

Remark 2.1. A function that is strongly differentiable as in cases (i) and (ii) of definition 2.9., will be referred as (i) - differentiable or as (ii) - differentiable, respectively.

Lemma 2.1. If $u(t) = (x(t), y(t), z(t), w(t))$ is a trapezoidal fuzzy number valued function, then (a) if u is (i) - differentiable (Hukuhara differentiable) then $u' = (x', y', z', w)$. (b) if u is (ii)–differentiable then $u' = (w', z', y', x)$.

3 Fourth Order Runge- Kutta Method for Solving Fuzzy Differential Equations

Let us consider the first order fuzzy ordinary differential equations of the form

$$\left.\begin{array}{l} f'(t) = f(t, y) \\ f(t_0) = f_0 \end{array}\right\} \tag{3.1}$$

Let the exact solution be $[F(t_n)]_\alpha = [\underline{F}(t_n;\alpha), \overline{F}(t_n;\alpha)]$ (3.2)

and $[f(t_n)]_\alpha = [\underline{f}(t_n;\alpha), \overline{f}(t_n;\alpha)]$ (3.3)

be the approximate solutions of equation (3.1) respectively.

By using fourth order Runge- Kutta method, approximate solution is calculated as follows,

$$[f(t_n)]_\alpha = [\underline{f}(t_n;\alpha), \overline{f}(t_n;\alpha)]$$

$$\underline{f}(t_{n+1};\alpha) = \underline{f}(t_n;\alpha) + \sum_{j=1}^{4} w_j k_{j,1}(t_n, f(t_n;\alpha))$$

$$\overline{f}(t_{n+1};\alpha) = \overline{f}(t_n;\alpha) + \sum_{j=1}^{4} w_j k_{j,2}(t_n, f(t_n;\alpha))$$

where the w_j' s are constants Then $k_{j,1}$and $k_{j,2}$ for j =1, 2, 3, 4 are defined as follow,

$$k_{1,1}(t_n, f(t_n;\alpha)) = \min h\left\{f(t_n, u) / u \in (\underline{f}(t_n;\alpha)), \overline{f}(t_n;\alpha)\right\}$$

$$k_{1,2}(t_n, f(t_n;\alpha)) = \max h\left\{f(t_n, u) / u \in (\underline{f}(t_n;\alpha)), \overline{f}(t_n;\alpha)\right\}$$

$$k_{2,1}(t_n, f(t_n;\alpha)) = \min h\left\{f(t_n + \frac{h}{2}, u) / u \in (p_{1,1}(t_n, f(t_n;\alpha), p_{1,2}(t_n, f(t_n;\alpha)\right\}$$

$$k_{2,2}(t_n, f(t_n;\alpha)) = \max h\left\{f(t_n + \frac{h}{2}, u) / u \in (p_{1,1}(t_n, f(t_n;\alpha), p_{1,2}(t_n, f(t_n;\alpha)\right\}$$

$$K_{3,1}(t_n, f(t_n;\alpha)) = \min\ h\left\{ f(t_n + \frac{h}{2}, u) / u \in (p_{2,1}(t_n, f(t_n;\alpha), p_{2,2}(t_n, f(t_n;\alpha) \right\}$$

$$k_{3,2}(t_n, f(t_n;\alpha)) = \max\ h\left\{ f(t_n + \frac{h}{2}, u) / u \in (p_{2,1}(t_n, f(t_n;\alpha), p_{2,2}(t_n, f(t_n;\alpha) \right\}$$

$$k_{4,1}(t_n, f(t_n;\alpha)) = \min\ h\left\{ f(t_n + \frac{h}{2}, u) / u \in (p_{3,1}(t_n, f(t_n;\alpha), p_{3,2}(t_n, f(t_n;\alpha) \right\}$$

$$k_{4,2}(t_n, f(t_n;\alpha)) = \max h\left\{ f(t_n + \frac{h}{2}, u) / u \in (p_{3,1}(t_n, f(t_n;\alpha), p_{3,2}(t_n, f(t_n;\alpha) \right\}$$

where $p_{1,1}(t_n, f(t_n;\alpha)) = \underline{f}(t_n;\alpha) + \frac{h}{2} k_{1,1}(t_n, f(t_n;\alpha)$

$$p_{1,2}(t_n, f(t_n;\alpha)) = \overline{f}(t_n;\alpha) + \frac{h}{2} k_{1,2}(t_n, f(t_n;\alpha)$$

$$p_{2,1}(t_n, f(t_n;\alpha)) = \underline{f}(t_n;\alpha) + \frac{h}{2} k_{2,1}(t_n, f(t_n;\alpha)$$

$$p_{2,2}(t_n, f(t_n;\alpha)) = \overline{f}(t_n;\alpha) + \frac{h}{2} k_{2,2}(t_n, f(t_n;\alpha)$$

$$p_{3,1}(t_n, f(t_n;\alpha)) = \underline{f}(t_n;\alpha) + \frac{h}{2} k_{3,1}(t_n, f(t_n;\alpha)$$

$$p_{3,2}(t_n, f(t_n;\alpha)) = \overline{f}(t_n;\alpha) + \frac{h}{2} k_{3,2}(t_n, f(t_n;\alpha)$$

Now, using the initial conditions x_0, y_0 and the fourth order Runge – Kutta formula, we compute,

$$\left.\begin{aligned} \underline{f}(t_{n+1};\alpha) &= \underline{f}(t_n;\alpha) + \frac{1}{6}(k_{1,1}(t_n, f(t_n;\alpha)) + 2k_{2,1}(t_n, f(t_n;\alpha)) + 2k_{3,1}(t_n, f(t_n;\alpha)) + k_{4,1}(t_n, f(t_n;\alpha))) \\ \overline{f}(t_{n+1};\alpha) &= \overline{f}(t_n;\alpha) + \frac{1}{6}(k_{1,2}(t_n, f(t_n;\alpha)) + 2k_{2,2}(t_n, f(t_n;\alpha)) + 2k_{3,2}(t_n, f(t_n;\alpha)) + k_{4,2}(t_n, f(t_n;\alpha))) \end{aligned}\right\} \quad (3.4)$$

The exact and approximate solutions at $t_{n,}\ 0 \le n \le N$ are denoted by $[F(t_n)]_\alpha = [\underline{F}(t_n;\alpha)\ \overline{F}(t_n;\alpha)]$ and $[f(t_n)]_\alpha = [\underline{f}(t_n;\alpha)\ \overline{f}(t_n;\alpha)]$ respectively. The solution is calculated by grid points $a = t_0 \le t_1 \le t_2 \le .... \le t_n = b$ and $h = \frac{(b-a)}{N} = t_{n+1} - t_n$.

$$\underline{F}(t_{n+1};\alpha) = \underline{F}(t_n;\alpha) + \frac{1}{6} F[t_n, f(t_n;\alpha)]$$

$$\overline{F}(t_{n+1};\alpha) = \overline{F}(t_n;\alpha) + \frac{1}{6} G[t_n, f(t_n;\alpha)] \text{ and}$$

$$\underline{f}(t_{n+1};\alpha) = \underline{f}(t_n;\alpha) + \frac{1}{6} F[t_n, f(t_n;\alpha)]$$

$$\overline{f}(t_{n+1};\alpha) = \overline{f}(t_n;\alpha) + \frac{1}{6} G[t_n, f(t_n;\alpha)]$$

The following lemmas will be applied to show the convergences of theses approximation $\lim_{h\to 0} \underline{f}(t,\alpha)=\underline{F}(t,\alpha)$ and $\lim_{h\to 0} \overline{f}(t,\alpha)=\overline{F}(t,\alpha)$.

Lemma 3.1. Let the sequence of number $\{w_n\}_{n=0}^{N}$ satisfy $|w_{n+1}| \le A|w_n| + B, 0 \le n \le N-1$ for some given positive constants A and B (proof [20]) then $|w_n| \le A^n |w_0| + B\frac{A^n-1}{A-1}, 0 \le n \le N-1$.

The proof of Lemma (3.1) follows Lemma 2 of Ming Ma, Kandel [9].

Lemma 3.2. Let the sequence of numbers $\{w_n\}_{n=0}^{N}$ $\{v_n\}_{n=0}^{N}$ satisfy $|w_{n+1}| \le |w_n| + A\max\{|w_n|,|v_n|\} + B$ $|v_{n+1}| \le |v_n| + A\max\{|w_n|,|v_n|\} + B$, for some given positive constants A and B, and denote $|u_n| = |w_n| + |v_n|, 0 \le n \le N$. Then $|u_n| \le \overline{A}^n |u_0| + \overline{B}\frac{\overline{A}^n-1}{\overline{A}-1}, 0 \le n \le N$, where $\overline{A} = 1+2A$ and $\overline{B} = 2B$.

Proof of Lemma (3.2) follows Lemma 3.1.

Theorem 3.1. Let $F[t,u,v]$, $G[t,u,v]$, belongs to $C^4(k)$ and let the partial derivatives of F, G be bounded over R Then for arbitrary fixed $r: 0 \le r \le 1,$ the approximate solutions converge uniformly in t to the exact solutions.

This theorem is simply proved (see proof theorem 4.1 in [10]).

4 Numerical Examples

Example 4.1. Consider the Fuzzy initial value problem,

$$\begin{cases} y'(t) = y(t), t \in [0,1] \\ y(0) = (0.8+0.125r, 1.1-0.1r), 0 < r \le 1 \end{cases}$$

The exact solution is obtained as

$$\underline{Y}(t;r) = \underline{y}(t;r)e^t,$$
$$\overline{Y}(t;r) = \overline{y}(t;r)e^t$$
$$\text{at } t=1, Y(1;r) = [(0.8+0.125r)e, (1.1-0.1r)e]$$

Table 1. Comparison between the exact solution and approximate solutions of [13] and the proposed method for t =1 by complete error analysis

r	t	Exact solutions at t =1		Approximated solutions at h=0.001 [13]		Approximated solutions at h=0.01 (Proposed method)	
		$\underline{Y}$	$\overline{Y}$	$\underline{y}$	$\overline{y}$	$\underline{y}$	$\overline{y}$
0	1	2.1746	2.9901	2.1746	2.9901	2.1746	2.9901
0.2	1	2.2425	2.9357	2.2425	2.9357	2.2425	2.9357
0.4	1	2.3105	2.8813	2.3105	2.8813	2.3105	2.8813
0.6	1	2.3784	2.8270	2.3784	2.8270	2.3784	2.8270
0.8	1	2.4464	2.7726	2.4464	2.7726	2.4464	2.7726
1	1	2.514	2.7182	2.5144	2.7182	2.5144	2.7182

5 Fuzzy Integration

Theorem 5.1 [24]. Let $f:(a,b)\to R$ and $t\in(a,b)$ be a fuzzy valued function and denote $f(t)=[\underline{f}(t,\alpha),\overline{f}(t,\alpha]$ for each $0\le\alpha\le 1$, the following conditions are true.

(i). If f is differentiable in the 1st form (i) in Definition 2.9, then $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are differentiable $f'(t)=\left[\underline{f}'(t,\alpha),\overline{f}'(t,\alpha)\right]$.

(ii). If f is differentiable in the 2nd form (ii) in Definition 2.9, then $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are differentiable $f'(t)=\left[\overline{f}'(t,\alpha),\underline{f}'(t,\alpha)\right]$.

Theorem 5.2. Let $f:(a,b)\to R$ and $t\in(a,b)$ be a fuzzy valued function and denote $f(t)=[\underline{f}(t,\alpha),\overline{f}(t,\alpha]$ for each $0\le\alpha\le 1$, Then

(i). If f and f' are differentiable in the 1st form (i) in Definition 2.9 or if f and f' are differentiable in the 2nd form (ii) Definition 2.9, then $\underline{f}'(t,\alpha)$ and $\overline{f}'(t,\alpha)$ are differentiable and $f''(t)=\left[\underline{f}''(t,\alpha),\overline{f}''(t,\alpha)\right]$.

(ii). If f is differentiable in the 1st form (i) f' is differentiable in the 2nd form(ii) or if f is differentiable in the 2nd form (ii) and f' is differentiable in the first form (i) in Definition 2.9 then $\underline{f}'(t,\alpha)$ and $\overline{f}'(t,\alpha)$ are differentiable and $f''(t)=\left[\overline{f}''(t,\alpha),\underline{f}''(t,\alpha)\right]$.

Theorem 5.3. Let $f(x)$ be a fuzzy real valued function on $[a,\infty)$ and it is represented by $\left[\underline{f}(x,\alpha),\overline{f}(x,\alpha)\right]$. For any fixed $r\in[0,1]$, assume $\underline{f}(x,\alpha),\overline{f}(x,\alpha)$ are Rimmann-intergrable on [a, b] for every $b\ge a$ and let there exists two positive $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ such that

$$\int_a^b|\underline{f}(x,\alpha)|dx\le\underline{M}(\alpha) \text{ and } \int_a^b|\overline{f}(x,\alpha)|dx\le\overline{M}(\alpha)$$

for every b ≥ a. Then $f(x)$ is improper fuzzy Rimmann-intergable on $[a,\infty)$ and the improper fuzzy Rimmann-intergable is a fuzzy number. Furthermore we have

$$\int_a^\infty f(x)dx = \left(\int_a^\infty \underline{f}(x,\alpha)dx, \int_a^\infty \overline{f}(x,\alpha)dx \right)$$

6 Simpson's Rule

Consider the solution curve y=y(t) in the left continuous over the first subinterval $[t_0,t_1]$.The function values in (3.4) are approximations for slopes to this curve. Here $k_{1,1}$ is the slope the left, $k_{2,1}$ *and* $k_{3,1}$ are two estimates for the slope in the middle, and $k_{4,1}$ is the slope at the right. The next point (t_1, y_1) is obtained by integrating the slope function.

In the same way we have the solution of right continuous curve y=y(t) over the first subinterval $[t_0,t_1]$.The function values in (3.4) are approximations for slopes to this curve. Here $k_{1,2}$ is the slope at the left $k_{2,2}$ and $k_{3,2}$ are two estimates for the slope in the middle, and right $k_{4,2}$ is the slope at the right. The next point (t_1, y_1) is obtained by integrating the slope function

$$\left.\begin{aligned} y(t_1) - y(t_0) &= \int_{t_0}^{t_1} \underline{f}(t, y(t))dt \\ y(t_1) - y(t_0) &= \int_{t_0}^{t_1} \overline{f}(t, y(t))dt \end{aligned}\right\} \quad (6.1)$$

If Simpson's rule is applied with step h/2, the approximation to the integral in (6.1) is

$$\left.\begin{aligned} \int_{t_0}^{t_1} \underline{f}(t, y(t))dt &\approx \frac{h}{6} [\underline{f}(t_0, y(t_0)) + 4\underline{f}(t_{1/2}, y(t_{1/2})) + \underline{f}(t_1, y(t_1))], \\ \int_{t_0}^{t_1} \overline{f}(t, y(t))dt &\approx \frac{h}{6} [\overline{f}(t_0, y(t_0)) + 4\overline{f}(t_{1/2}, y(t_{1/2})) + \overline{f}(t_1, y(t_1))], \end{aligned}\right\} \quad (6.2)$$

where $t_{1/2}$ is the midpoint of the interval. Three function values are needed; hence we make the obvious choice $\underline{f}(t_0, y(t_0)) = k_{1,1}$ and $f(t_1, y(t_1)) = k_{4,1}$ for the value in the middle we choose the average of $k_{2,1}$ *and* $k_{3,1}$

$$f(t_{1/2}, y(t_{1/2})) \approx \frac{k_{2,1} + k_{3,1}}{2} \qquad \text{and}$$

$$f(t_{1/2}, y(t_{1/2})) \approx \frac{k_{2,2} + k_{3,2}}{2}$$

This value is substituted in (6.2), which is used in equation (6.1).

$$y_1 = y_0 + \frac{h}{6}(k_{1,1} + \frac{4(k_{2,1} + k_{3,1})}{2} + k_{4,1},\ k_{1,2} + \frac{4(k_{2,2} + k_{3,2})}{2} + k_{4,2})$$

Remark 6.1. Even though RK method of 4th order is easy to determine the accuracy and also RK4 solution has been computed, we can use Simpson's rule of step size h/2 for better accuracy. Instead of using Runge - kutta method for the study of real system we can choose the solution of simpson's rule which reflects the better behaviour of the system. To get complete error analysis, Runge - kutta method of order 4 requires a small step size h=0.01, but Simpson's rule requires of step size h=0.1 over [0, 1]. When the step size decreases, we have to repeat the algorithm to get the given interval. These advantages are shown in the following simple examples.

Example 6.1. Consider the Fuzzy initial value problem involving trapezoidal fuzzy numbers,

$$\begin{cases} y'(t) = y(t), t \in [0,1] \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r), 0 < r \leq 1 \end{cases}$$

Comparison between the exact solution and approximate solution of Simpson's Rule for t =1 by complete error analysis.

Table 2. Comparison between the exact solution and approximate solution obtained by the proposed method (Simpson's Rule) for t =1 by complete error analysis

r	t	Exact solutions at t =1		Approximated solutions at h=0.1 Simpson's Rule	
		$\underline{Y}$	$\overline{Y}$	$\underline{y}$	$\overline{y}$
0	1	2.174625	2.990110	2.174625	2.990110
0.2	1	2.242583	2.935744	2.242583	2.935744
0.4	1	2.310540	2.881379	2.310540	2.881379
0.6	1	2.378497	2.827013	2.378497	2.827013
0.8	1	2.446454	2.772647	2.446454	2.772647
1	1	2.514411	2.718282	2.514411	2.718282

Table 3. Comparison between the exact solution and approximate solution obtained by the proposed method (Simpson's Rule) for t =1 by complete error analysis

r	t	Exact solutions at t =1		Approximated solutions at h=0.1 Simpson's Rule	
		$\underline{Y}$	$\overline{Y}$	$\underline{y}$	$\overline{y}$
0	1	-2.7183	0	-2.7183	0
0.2	1	-2.1746	0.5436	-2.1746	0.5436
0.4	1	-1.6310	1.0873	-1.6310	1.0873
0.6	1	-1.0873	1.6310	-1.0873	1.6310
0.8	1	-0.5436	2.1746	-0.5436	2.1746
1	1	0	2.7183	0	2.7183

Example 6.2. Consider the following fuzzy differential equation with fuzzy initial value problem involving triangular fuzzy numbers,

$$y'(t) = -\lambda \Theta y(t), \text{ with } y(0) = y_0 = [\alpha - 1,\ 1 - \alpha].$$

We get the exact solution as $y(t,\alpha)=[(\alpha-1)e^{t},(1-\alpha)e^{t}]$

Divide the interval [0,1] (t =1) into n=10 equally spaced subintervals. By applying the generalized Simpson's rule, we obtain the approximate solution of the given problem for t =1 by complete error analysis

7 Conclusions

In this paper, we have proposed Runge Kutta method of 4th order and Simpson's rule for finding numerical solution of fuzzy differential equations involving triangular and trapezoidal fuzzy numbers by complete error analysis. From the above examples, we see that the solution of fuzzy differential equations obtained by Runge Kutta method of 2nd and 4th orders and Simpson's rule coincide with the exact solution but only the step size varies. To get complete error analysis, Runge-Kutta method of order 4 requires a small step size h=0.01, but Simpson's rule requires step size h=0.1 over [0, 1]. The trickiest part of using Runge-Kutta methods to approximate the solution of a differential equation is choosing the right step-size. Too large a step-size and the error is too large and the approximation is inaccurate. Too small a step-size and the process will take too long and possibly have too much round off error to be accurate. Furthermore, the appropriate step-size may change during the course of a single problem. Many problems in celestial mechanics, chemical reaction kinematics, and other areas have long periods of time where nothing much is happening (and for which large step-sizes are appropriate) mixed in with periods of intense activity where a small step-size is vital. What we need is an algorithm which includes a method for choosing the appropriate step-size at each step. This is the main disadvantage of Runge-Kutta method.

In future we can develop this work with higher order fuzzy differential equation and system of fuzzy linear differential equation.

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Competing Interests

Authors have declared that no competing interests exist.

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