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Properties and Characterizations of Countably C-approximating Posets

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Abstract

In this paper, the concept of countably C-approximating posets is introduced. Properties and characterizations of countably C-approximating posets are presented. Main results are: (1) the lattice of all σ-Scott-closed subsets for any poset is countably C-approximating; (2) a complete lattice is completely distributive iff it is countably approximating and countably C-approximating.

Keywords: σ-Scott topology; countably C-approximating poset; semilattice; principal ideal; completely distributive lattice.

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1 Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [1]. Later, a more general notion of continuous directed complete partially

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ordered sets (in short, continuous dcpos or domains) was introduced and extensively studied (see [2]-[4]). Since many naturally arisen posets are important but fail to be directed complete partially ordered sets (in short, dcpos), there are more and more occasions to study posets which miss suprema of directed sets (see [5]-[10], [12]). Lawson in [3] gave a remarkable characterization that a dcpo P is continuous iff the lattice $\sigma^*(P)$ of all Scott-closed subsets of P is completely distributive. By the technique of embedded bases and sobrification via the Scott topology, Xu in [6] successfully embedded continuous posets into continuous dcpos and proved that a poset P is continuous iff $\sigma^*(P)$ is completely distributive. In order to study the order structure of $\sigma^*(P)$ of a non-continuous poset P, Ho and Zhao in [7] introduced the concept of C-continuous posets. They showed that $\sigma^*(P)$ is a C-continuous lattice for any poset P and that a complete lattice is completely distributive if and only if it is continuous and C-continuous.

On the other hand, Lee in [8] introduced the concept of countably approximating lattices, a generalization of continuous lattices and showed that this new larger class has many properties in common with continuous lattices. In [9], Han, Hong, Lee and Park further generalized the concept of countably approximating lattices to the concept of countably approximating posets and characterized countably approximating posets via the σ-Scott topology.

In this paper, making use of the ideas of [7] and [9], we introduce the concept of countably C-approximating posets and discuss characterizations and properties of countably C-approximating posets. We will show that the lattice of all σ-Scott-closed subsets of a poset is a countably C-approximating lattice, and that a complete lattice is completely distributive if and only if it is countably approximating and countably C-approximating.

2 Preliminaries

We quickly recall some basic notions and results (see, for example, [4], [7] or [9]). Let $(P, \leq)$ be a poset. Then P with the dual order is also a poset and denoted by P^*. A *principal ideal* is a set of the form $\downarrow x = \{y \in P \mid y \leq x\}$. For $X \subseteq P$, we write $\downarrow X = \{y \in P \mid \exists\, x \in X, y \leq x\}$, $\uparrow X = \{y \in P \mid \exists\, x \in X, x \leq y\}$. A subset X is a(n) *lower set* (resp., *upper set*) if $X = \downarrow X$ (resp., $X = \uparrow X$). The supremum of X is denoted by $\vee X$ or $\sup X$. The notation $\sup_b X$ denotes the supremum of the subset $X \subseteq \downarrow b$ in the principal ideal $\downarrow b$. A nonempty subset D of P is *directed* if x, $y \in D$ implies there exists $z \in D$ with $x \leq z$ and $y \leq z$. A subset D is *countably directed* if every countable subset of D has an upper bound in D. Clearly every countably directed set is directed but not vice versa. A poset P is a *directed complete partially ordered set* (dcpo, in short) if every directed subset of P has a supremum. A poset is said to have *countably directed joins* if every countably directed subset has a supremum.

It is clear that if D is countably directed and itself is countable, then D has a maximal element. By this observation, we see that every countable poset has countably directed joins and thus a poset having countably directed joins needn't be a dcpo.

The following definitions give several induced relations by the order of a poset.

Definition 2.1. (see [4,6]) Let P be a poset and x, $y \in P$. We say that x *approximates* y, written $x \ll y$ if whenever D is a directed set that has a supremum $\sup D \geq y$, then there is some $d \in D$ with $x \leq d$. A poset is said to be *continuous* if every element is the directed supremum of elements that approximate it. A continuous poset which is also a complete lattice is called a *continuous lattice*.

Definition 2.2. (see [9]) Let P be a poset and x, $y \in P$. We say that x is *countably way below* y, written $x \ll_c y$ if for any countably directed subset D of P with $\sup D \geq y$, there is some $d \in D$ with $x \leq d$. For each $x \in P$, we write $\Downarrow_c x = \{y \in P \mid y \ll_c x\}$ and $\Uparrow_c x = \{y \in P \mid x \ll_c y\}$. A poset P having countably directed joins is called *a countably approximating poset* if for each $x \in P$, the set $\Downarrow_c x$ is countably directed and $\sup \Downarrow_c x = x$. A countably approximating poset which is also a complete lattice is called a *countably approximating lattice*.

Example 2.3. *Let P be the unit interval $[0, 1]$. For all $x \in [0, 1]$, it is easy to check that $\Downarrow_c x = \downarrow x$.*

In a poset P, since every countably directed set is directed, we have that $x \ll y$ implies $x \ll_c y$ for all $x, y \in P$. However, by Example 2.3, the reverse implication need not be true.

Since every countably directed subset of a countable poset has a maximal element, every countable poset is a countably approximating poset.

The proof of the following proposition is straightforward and is omitted.

Proposition 2.4. *Let P be a poset and S a countably subset of P such that $\vee S$ exists. If $s \ll_c x$ for all $s \in S$, then $\vee S \ll_c x$.*

By Proposition 2.4, in a complete lattice P, the set $\Downarrow_c x$ is countably directed for each $x \in P$. So, a complete lattice P is countably approximating if and only if for each $x \in P$, $x = \sup \Downarrow_c x$. Thus every continuous lattice is a countably approximating lattice.

As a generalization of completely distributive lattices, the following concept of supercontinuous posets was introduced in [10].

Definition 2.5. (see [10]) Let P be a poset and $x, y \in P$. We write $x \triangleleft y$ if for any subset $A \subseteq P$ for which $\vee A$ exists, $\vee A \geq y$ always implies that there exists $z \in A$ with $x \leq z$. A poset P is called *supercontinuous* if for each $a \in P$, $a = \vee\{x \in P \mid x \triangleleft a\}$.

It is clear that for all $x, y \in P$, $x \triangleleft y$ implies that $x \ll y$ and $x \ll_c y$.

Definition 2.6. (see [4, 6]) A subset U of a poset P is *Scott-open* if $\uparrow U = U$ and for any directed set $D \subseteq P$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. All the Scott-open sets of P form a topology, called the *Scott topology* and denoted by $\sigma(P)$. The complement of a Scott-open set is called a *Scott-closed* set. The collection of all Scott-closed sets of P is denoted by $\sigma^*(P)$.

Replacing directed sets with countably directed sets in Definition 2.6, we get the concept of σ-Scott-open sets.

Definition 2.7. (see [9]) Let P be a poset. A subset U of P is called *σ-Scott-open* if $\uparrow U = U$ and for any countably directed set $D \subseteq P$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. All the σ-Scott-open sets of P form a topology, called the *σ-Scott topology* and denoted by $\sigma_c(P)$. The complement of a σ-Scott-open set is called a *σ-Scott-closed* set. The collection of all σ-Scott-closed sets of P is denoted by $\sigma_c^*(P)$.

Remark 2.8. (see [9, Remark 2.1]) (1) For a poset P, the σ-Scott topology $\sigma_c(P)$ is closed under countably intersections and the Scott topology $\sigma(P)$ is coarser than $\sigma_c(P)$, i.e., $\sigma(P) \subseteq \sigma_c(P)$.

(2) A subset of a poset is σ-Scott-closed if and only if it is a lower set and closed under countably directed joins.

Definition 2.9. (see [9]) A function $f : P \to Q$ between posets P and Q is called *σ-Scott-continuous* if it is continuous with respect to the σ-Scott topologies on P and Q.

Proposition 2.10. (see [9, Remark 2.1]) A function $f : P \to Q$ between posets P and Q is σ-Scott-continuous if and only if it is order-preserving and $f(\sup D) = \sup f(D)$ whenever D is a countably directed set in P for which $\sup D$ exists.

Ho and Zhao in [7] introduced the concept of C-continuous posets via the Scott topology on posets.

Definition 2.11. (see [7]) Let P be a poset and $x, y \in P$. We say that x is *beneath* y, denoted by $x \prec y$, if for any nonempty Scott-closed set $F \subseteq P$ for which $\vee F$ exists, $\vee F \geq y$ always implies that $x \in F$. The poset P is said to be C*-continuous* if for each $x \in P$, $x = \vee \downarrow^{\prec} x$, where the set $\downarrow^{\prec} x = \{y \in P \mid y \prec x\}$. A C-continuous poset which is also a complete lattice is called a C*-continuous lattice*.

3 Countably C-approximating Posets

In this section, we define a new auxiliary relation on a poset and introduce the concept of countably C-approximating posets. We also present some properties and characterizations of countably C-approximating posets.

Definition 3.1. *Let P be a poset and $x, y \in P$. We say that x is* σ-beneath y*, denoted by $x \prec_\sigma y$, if for any nonempty σ-Scott-closed set $F \subseteq P$ for which $\vee F$ exists, $\vee F \geq y$ always implies that $x \in F$.*

The following proposition shows that the relation $\prec_\sigma$ on a poset P is indeed an auxiliary order by [4, Definition I-1.11].

Proposition 3.2. *For a poset P and $x, y, u, v \in P$, the following statements hold:*
(i) $x \lhd y \Rightarrow x \prec_\sigma y \Rightarrow x \prec y \Rightarrow x \leq y$;
(ii) $u \leq x \prec_\sigma y \leq v \Rightarrow u \prec_\sigma v$;
(iii) $\perp \prec_\sigma x$ *whenever P has a smallest element $\perp$.*

Proof. (i) Follows from Definitions 2.11, 2.5, 3.1 and Remark 2.8 (1).
(ii) and (iii) Straightforward. □

Proposition 3.3. *Let P be a poset and D a countably directed subset of P such that $\vee D$ exists. If $d \prec_\sigma x$ for all $d \in D$, then $\vee D \prec_\sigma x$.*

Proof. Let $F \in \sigma_c^*(P)$ be nonempty such that $\vee F$ exists with $\vee F \geq x$. Since $d \prec_\sigma x$ for all $d \in D$, it follows from Definition 3.1 that $d \in F$ and thus $D \subseteq F$. Because F is σ-Scott-closed and D is countably directed, we have $\vee D \in F$ by Remark 2.8 (2). This shows that $\vee D \prec_\sigma x$. □

By Remark 2.8 (2), Proposition 3.2 (ii) and Proposition 3.3, we immediately have the following corollary.

Corollary 3.4. *Let P be a poset. Then for all $x \in P$, the set $\downarrow^{\prec_\sigma} x = \{y \in P \mid y \prec_\sigma x\}$ is a σ-Scott-closed subset of P.*

With the relation $\prec_\sigma$ on a poset, we now introduce the concept of countably C-approximating posets.

Definition 3.5. *A poset P is said to be* countably C-approximating *if for each $x \in P$, $x = \vee \downarrow^{\prec_\sigma} x$, where the set $\downarrow^{\prec_\sigma} x = \{y \in P \mid y \prec_\sigma x\}$. A countably C-approximating poset which is also a complete lattice is called a* countably C-approximating lattice*.*

Proposition 3.6. *Every countably C-approximating poset is C-continuous.*

Proof. Let P be a countably C-approximating poset. Then for each $x \in P$, it follows from Proposition 3.2 (i) and Definition 3.5 that $\downarrow^{\prec_\sigma} x \subseteq \downarrow^{\prec} x$ and $x = \vee \downarrow^{\prec_\sigma} x$. Thus $x = \vee \downarrow^{\prec} x$. By Definition 2.11, P is C-continuous. □

The *lifting* of a poset P, denoted by $P_\perp$, is the poset obtained from P by adjoining a new bottom element. In the sequel, we give some propositions which characterize countably C-approximating posets by the σ-Scott topology, the lifting and principal ideals of the posets, respectively.

Proposition 3.7. *For a poset P, the following conditions are equivalent:*
(1) *P is countably C-approximating;*
(2) *$\forall\, x \in P$, the set $\downarrow^{\prec_\sigma} x$ is the smallest nonempty σ-Scott-closed subset F with $\vee F \geq x$;*
(3) *$\forall\, x \in P$, there is a smallest nonempty σ-Scott-closed subset F with $\vee F \geq x$.*

Proof. $(1) \Rightarrow (2)$: Suppose P is countably C-approximating and $x \in P$. By Corollary 3.4 and Definition 3.5, the set $\downarrow^{\prec_\sigma} x$ is a nonempty σ-Scott-closed subset of P with $\vee \downarrow^{\prec_\sigma} x \geq x$. Let F be another σ-Scott-closed subset with $\vee F \geq x$. For each $y \in \downarrow^{\prec_\sigma} x$, it follows from Definition 3.1 that $y \in F$. Thus the set $\downarrow^{\prec_\sigma} x \subseteq F$. This shows that the set $\downarrow^{\prec_\sigma} x$ is the smallest nonempty σ-Scott-closed set F with $\vee F \geq x$.

$(2) \Rightarrow (3)$: Trivial.

$(3) \Rightarrow (1)$: For each $x \in P$, let F_x be the smallest nonempty σ-Scott-closed subset F with $\vee F \geq x$. Then for all $F \in \sigma_c^*(P)$ with existing $\vee F \geq x$, we have $F_x \subseteq F$. It follows from Definition 3.1 that $t \prec_\sigma x$ for all $t \in F_x$. This shows that $F_x \subseteq \downarrow^{\prec_\sigma} x$. Clearly, x is an upper bound of the set $\downarrow^{\prec_\sigma} x$. Suppose that z is any upper bound of $\downarrow^{\prec_\sigma} x$. It follows from $F_x \subseteq \downarrow^{\prec_\sigma} x$ that z is also an upper bound of F_x. Thus $x \leq \vee F_x \leq z$. This shows that x is the least upper bound of $\downarrow^{\prec_\sigma} x$ and hence $x = \vee \downarrow^{\prec_\sigma} x$. By Definition 3.5, P is countably C-approximating. $\Box$

Proposition 3.8. *A poset P is countably C-approximating iff $P_\perp$ is countably C-approximating.*

Proof. Suppose that P is countably C-approximating and $x \in P_\perp$. If $x \in P$, then by Proposition 3.7, there is a smallest nonempty σ-Scott-closed subset F_x of P with $\vee F_x \geq x$. Thus $F'_x = F_x \cup \{\perp\}$ is the smallest nonempty σ-Scott-closed subset of $P_\perp$ with $\vee F'_x \geq x$. If $x = \perp$, then by Proposition 3.2 (iii), $F_\perp = \{\perp\}$ is the smallest nonempty σ-Scott-closed subset of $P_\perp$ with $\vee F_\perp \geq \perp$. Thus by Proposition 3.7, $P_\perp$ is countably C-approximating.

Conversely, suppose that $P_\perp$ is countably C-approximating and $x \in P$. By Proposition 3.7, there is a smallest nonempty σ-Scott-closed subset F'_x of $P_\perp$ with $\vee F'_x \geq x$. Then $F_x = F'_x \setminus \{\perp\}$ is the smallest nonempty σ-Scott-closed subset F_x of P with $\vee F_x \geq x$. Thus by Proposition 3.7 again, P is countably C-approximating. $\Box$

To characterize countably C-approximating posets by principal ideals, we need some new concepts and results.

Lemma 3.9. *Let P be a poset, $x \in P$ and $A \subseteq \downarrow x = \varphi$. Then $\vee_\varphi A = \vee A$ whenever $\vee A$ exists; If P is a semilattice, then $\vee A = \vee_\varphi A$ whenever $\vee_\varphi A$ exists, where $\vee_\varphi A$ denotes the supremum of the subset A in the principal ideal $\varphi = \downarrow x$.*

Proof. Straightforward. $\Box$

Definition 3.10. *The σ-Scott topology on a poset P is called* lower hereditary *if for every σ-Scott-closed subset A, the relative σ-Scott topology on A agrees with the σ-Scott topology of the poset A in the hereditary order of P.*

Lemma 3.11. *Let P be a poset. The following statements are equivalent:*
(1) *The σ-Scott topology on P is lower hereditary;*
(2) *For any $x \in P$, the inclusion map from the poset $\downarrow x$ into P is σ-Scott-continuous;*
(3) *Any minimal upper bound of any countably directed set in P is a (the) least upper bound for that countably directed set.*

Proof. $(1) \Rightarrow (2)$: Follows from that any principal ideal is a σ-Scott-closed set.

$(2) \Rightarrow (3)$: Let D be a countably directed set with minimal upper bound b. Then b is the supremum of D in $\downarrow b$. Since the inclusion map of $\downarrow b$ into P is σ-Scott-continuous, it follows from Proposition 2.10 that b is the supremum of D in P.

$(3) \Rightarrow (1)$: Let E be a σ-Scott-closed set. A subset B that is σ-Scott-closed in E is easily verified to be σ-Scott-closed in P. Conversely suppose that A is σ-Scott-closed in P. Then $A \cap E$ is a lower set. To show that $A \cap E$ is σ-Scott-closed in E, let D be a countably directed set in $A \cap E$ that has supremum b in E. Then $\sup_b D = b \in E$ and by (3), $b = \sup D$. It follows from σ-Scott-closedness of A that $b \in A$. Hence $b \in A \cap E$. This shows that $A \cap E$ is closed in the σ-Scott topology of E. $\square$

Applying Lemmas 3.9 and 3.11 (3), we obtain the following corollary.

Corollary 3.12. *Every semilattice has a lower hereditary σ-Scott topology.*

Proposition 3.13. *Let P be a semilattice. If P is countably C-approximating, then every principal ideal of P is countably C-approximating.*

Proof. Suppose P is a countably C-approximating semilattice. Then we claim that for all $x \in P$ and $u \in \downarrow x = \varphi$, $\downarrow^{\prec_\sigma} u \subseteq \downarrow_\varphi^{\prec_\sigma} u$ holds. In fact, for all $a \in \downarrow^{\prec_\sigma} u$ and for all $A \in \sigma_c^*(\downarrow x)$ with existing $\vee_\varphi A \geq u$, by Definition 3.10 and Corollary 3.12, we have $A \in \sigma_c^*(P)$. It follows from Lemma 3.9 that $x \geq \vee_\varphi A = \vee A \geq u$. Since $a \in \downarrow^{\prec_\sigma} u$, there is $v \in A \subseteq \downarrow x$ such that $v \geq a$. This shows that $a \in \downarrow_\varphi^{\prec_\sigma} u$ and hence $\downarrow^{\prec_\sigma} u \subseteq \downarrow_\varphi^{\prec_\sigma} u$.

By the countably C-approximating property of P and Lemma 3.9, $u = \vee \downarrow^{\prec_\sigma} u = \vee_\varphi \downarrow^{\prec_\sigma} u$. Clearly, u is an upper bound of the set $\downarrow_\varphi^{\prec_\sigma} u$ in the principal ideal $\varphi = \downarrow x$. Suppose that t is any upper bound of $\downarrow_\varphi^{\prec_\sigma} u$ in φ. Then t is an upper bound of the set $\downarrow^{\prec_\sigma} u$ in φ. Thus $u = \vee \downarrow^{\prec_\sigma} u \leq t$. This shows that $\vee_\varphi \downarrow_\varphi^{\prec_\sigma} u = u$. Thus for all $x \in P$, the principal ideal $\varphi = \downarrow x$ is countably C-approximating. $\square$

Proposition 3.14. *Let P be a semilattice. If every principal ideal of P is countably C-approximating, then P is countably C-approximating.*

Proof. Let P be a semilattice. Suppose that for all $x \in P$, the principal ideal $\varphi = \downarrow x$ is countably C-approximating. We claim that $\downarrow_\varphi^{\prec_\sigma} x \subseteq \downarrow^{\prec_\sigma} x$. For all $y \in \downarrow_\varphi^{\prec_\sigma} x$ and for all $A \in \sigma_c^*(P)$ with existing $\vee A = z \geq x$, the principal ideal $\rho := \downarrow z$ by the hypothesis is countably C-approximating. By Corollary 3.4 and Corollary 3.12, $\downarrow_\rho^{\prec_\sigma} x \in \sigma_c^*(\downarrow z)$ and thus $\downarrow_\rho^{\prec_\sigma} x \in \sigma_c^*(\downarrow x)$. By Lemma 3.9, we have $\vee_\rho \downarrow_\rho^{\prec_\sigma} x = \vee_\varphi \downarrow_\rho^{\prec_\sigma} x = x$. Since $y \in \downarrow_\varphi^{\prec_\sigma} x$, there is $u \in \downarrow_\rho^{\prec_\sigma} x$ such that $y \leq u$. Since $A \in \sigma_c^*(P)$ with existing $\vee A = z \geq x$, by Corollary 3.4 and Corollary 3.12 again, we have $A \in \sigma_c^*(\downarrow z)$ and $\vee_\rho A = \vee A = z \geq x$. It follows from $u \in \downarrow_\rho^{\prec_\sigma} x$ that there is $a \in A$ such that $a \geq u \geq y$. This shows that $y \in \downarrow^{\prec_\sigma} x$ and hence $\downarrow_\varphi^{\prec_\sigma} x \subseteq \downarrow^{\prec_\sigma} x$.

By the countably C-approximating property of $\varphi = \downarrow x$ and Lemma 3.9, we have $x = \vee_\varphi \downarrow_\varphi^{\prec_\sigma} x = \vee \downarrow_\varphi^{\prec_\sigma} x$. Clearly x is an upper bound of $\downarrow^{\prec_\sigma} x$ in P. Suppose that t is any upper bound of $\downarrow^{\prec_\sigma} x$ in P. Then t is an upper bound of $\downarrow_\varphi^{\prec_\sigma} x$ in P. Thus $x = \vee \downarrow_\varphi^{\prec_\sigma} x \leq t$. This shows that $\vee \downarrow^{\prec_\sigma} x = x$ and hence P is countably C-approximating. $\square$

By Propositions 3.13 and 3.14, we immediately have the following characterization of countably C-approximating posets by principal ideals.

Theorem 3.15. *Let P be a semilattice. Then P is countably C-approximating if and only if every principal ideal of P is countably C-approximating.*

4 Countably C-approximating Lattices and CD-lattices

In this section, we explore relationships between countably C-approximating lattices and completely distributive lattices (CD-lattices, in short).

Proposition 3.10 in [7] shows that every C-continuous lattice is distributive. By Proposition 3.6, every countably C-approximating lattice is C-continuous. So, every countably C-approximating lattice is also distributive. In fact, countably C-approximating lattices, as special cases of C-continuous lattices, enjoy stronger distributivity.

Proposition 4.1. *Let P be a countably C-approximating lattice. Then for any collection $\{F_i \mid i \in I\}$ of nonempty countable subsets of P, the following equation holds:*

$$\bigwedge\{\bigvee F_i \mid i \in I\} = \bigvee\{\bigwedge\{f(i) \mid i \in I\} \mid f \in \Pi_{i\in I}F_i\}$$

.

Proof. Let P be a countably C-approximating lattice. For any collection $\{F_i \mid i \in I\}$ of nonempty countable subsets of P, let $a = \bigwedge\{\bigvee F_i \mid i \in I\}$ and $b = \bigvee\{\bigwedge\{f(i) \mid i \in I\} \mid f \in \Pi_{i\in I}F_i\}$. To show the equation holds, it suffices to prove that $a \leq b$. Suppose $t \in \downarrow^{\prec_\sigma} a$, then for each $i \in I$, $t \prec_\sigma a \leq \bigvee F_i = \bigvee \downarrow F_i$. It is easy to verify that $\downarrow F_i$ is σ-Scott-closed by Remark 2.8 (1). Thus $t \in \downarrow F_i$ and hence there is $d_i \in F_i$ with $t \leq d_i$ for each $i \in I$. Let $f \in \Pi_{i\in I}F_i$ be defined by $f(i) = d_i$, $i \in I$. Then $t \leq \bigwedge\{f(i) \mid i \in I\} \leq b$. By the countably C-approximating property of P, we have $a = \vee \downarrow^{\prec_\sigma} a$ and thus $a \leq b$. □

To characterize completely distributive lattices by the countably C-approximating property, we need the following lemma established in [11].

Lemma 4.2. (see [11, Theorem 2.6]) Let P be a complete lattice. Then P is completely distributive if and only if P is supercontinuous.

Proposition 4.3. *Every completely distributive lattice is countably C-approximating.*

Proof. Let P be a completely distributive lattice. For each $x \in P$, it follows from Definition 2.5, Lemma 4.2 and Proposition 3.2 (i) that $x = \vee \Downarrow^{\lhd} x$ and that $\Downarrow^{\lhd} x \subseteq \downarrow^{\prec_\sigma} x$. Thus $x = \vee \downarrow^{\prec_\sigma} x$. By Definition 3.5, P is a countably C-approximating lattice. □

Theorem 4.4. *Let P be a complete lattice. The following statements are equivalent:*
(1) *P is completely distributive;*
(2) *P is countably C-approximating and countably approximating.*

Proof. $(1) \Rightarrow (2)$: Follows from Proposition 4.3 and that every completely distributive lattice is a continuous lattice and hence a countably approximating lattice.

$(2) \Rightarrow (1)$: Suppose that P is countably C-approximating and countably approximating. For each $a \in P$, since P is countably approximating, we have $a = \vee \Downarrow_c a = \vee\{x \in P \mid x \ll_c a\}$. For each $x \in \Downarrow_c a$, since P is countably C-approximating, we have $x = \vee\{y \in P \mid y \prec_\sigma x\}$. Thus $a = \vee\{y \in P \mid \exists\, x$ such that $y \prec_\sigma x \ll_c a\}$. Suppose $y \prec_\sigma x \ll_c a$. Then we shall show that $y \lhd a$. For any $A \subseteq P$ with $\vee A \geq a$, let $D = \{\vee S \mid S$ is a countable subset of $A\}$. It is easy to verify that D is a countably directed set and $\vee D = \vee A \geq a$. Since $x \ll_c a$, there is a countable subset $S \subseteq A$ such that $x \leq \vee S = \vee \downarrow S$. By Remark 2.8 (1), $\downarrow S$ is σ-Scott-closed. It follows from $y \prec_\sigma x$ that $y \leq s$ for some $s \in S \subseteq A$. This shows that $y \lhd a$. Thus $a = \vee\{y \in P \mid \exists\, x$ such that $y \prec_\sigma x \ll_c a\} \leq \vee\{x \in P \mid x \lhd a\} \leq a$ and hence $a = \vee\{x \in P \mid x \lhd a\}$. By Definition 2.5 and Lemma 4.2, P is completely distributive. □

5 Countably C-approximating Property of $\sigma_c^*(P)$

In this section, we prove that the lattice of all σ-Scott-closed subsets for any poset is countably C-approximating.

Proposition 5.1. *Let P be a poset and $\mathcal{C} \in \sigma_c^*(\sigma_c^*(P))$. Then $\bigvee_{\sigma_c^*(P)} \mathcal{C} = \bigcup \mathcal{C}$.*

Proof. Note that each member of $\mathcal{C}$ is a σ-Scott-closed subset of P. So it suffices to show that $\bigcup \mathcal{C} \in \sigma_c^*(P)$. Clearly $\bigcup \mathcal{C}$ is a lower set. Let $D \subseteq \bigcup \mathcal{C}$ be any countably directed subset of P such that $\vee D$ exits in P. We prove that $\vee D \in \bigcup \mathcal{C}$. Construct $\mathcal{D} = \{\downarrow d \mid d \in D\}$. It is easy to verify that $\mathcal{D}$ is a countably directed subset of $\sigma_c^*(P)$ and $\bigvee_{\sigma_c^*(P)} \mathcal{D} = \downarrow \vee D$. Since $D \subseteq \bigcup \mathcal{C}$, for each $d \in D$, there is $C \in \mathcal{C}$ such that $\downarrow d \subseteq C$. This shows that $\mathcal{D} \subseteq \mathcal{C}$ because $\mathcal{C}$ is a lower set in $\sigma_c^*(P)$. Since $\mathcal{C}$ is a σ-Scott-closed subset of $\sigma_c^*(P)$ and $\mathcal{D} \subseteq \mathcal{C}$ is a countably directed subset of $\sigma_c^*(P)$, by Remark 2.8 (2) we have $\bigvee_{\sigma_c^*(P)} \mathcal{D} = \downarrow \vee D \in \mathcal{C}$. Thus $\vee D \in \bigcup \mathcal{C}$. By Remark 2.8 (2) again, $\bigcup \mathcal{C} \in \sigma_c^*(P)$. ☐

Definition 5.2. *An element k of a poset P is called* countably C-compact *if $k \prec_\sigma k$. The set of all countably C-compact elements of P is denoted by $\kappa_c(P)$.*

Proposition 5.3. *Let P be a poset and F be a nonempty σ-Scott-closed subset of P. Then for each $x \in F$, $\downarrow x \prec_\sigma F$ holds in $\sigma_c^*(P)$.*

Proof. Let F be a nonempty σ-Scott-closed subset of P. Then for each $x \in F$, suppose $\mathcal{C} \in \sigma_c^*(\sigma_c^*(P))$ with $\bigvee_{\sigma_c^*(P)} \mathcal{C} \supseteq F$. By Proposition 5.1, $\bigvee_{\sigma_c^*(P)} \mathcal{C} = \bigcup \mathcal{C} \supseteq F$. So there exists $C \in \mathcal{C}$ such that $x \in C$. Since $C \in \sigma_c^*(P)$ is a lower set, we have $\downarrow x \subseteq C$. It follows from $\mathcal{C} \in \sigma_c^*(\sigma_c^*(P))$ that $\downarrow x \in \mathcal{C}$. Thus by Definition 3.1, $\downarrow x \prec_\sigma F$ holds in $\sigma_c^*(P)$. ☐

By Proposition 5.3, we immediately have the following corollary.

Corollary 5.4. *Let P be a poset. Then for each $x \in P$, $\downarrow x \in \kappa_c(\sigma_c^*(P))$.*

Definition 5.5. *A poset P is said to be* countably C-prealgebraic *if for each $x \in P$, $x = \vee\{k \in \kappa_c(P) \mid k \leq x\}$. A countably C-prealgebraic poset which is also a complete lattice is called a* countably C-prealgebraic lattice.

It is easy to check that countably C-prealgebraic posets are countably C-approximating and thus countably C-prealgebraic lattices are all countably C-approximating lattices.

Now we arrive at our promised theorem of this section.

Theorem 5.6. *For any poset P, the lattice $\sigma_c^*(P)$ is a countably C-prealgebraic lattice, especially a countably C-approximating lattice.*

Proof. For each $F \in \sigma_c^*(P)$, it is straightforward to show that $F = \bigvee_{\sigma_c^*(P)} \{\downarrow x \mid x \in F\}$. By Corollary 5.4 and Definition 5.5, the lattice $\sigma_c^*(P)$ is a countably C-prealgebraic lattice. ☐

6 Conclusion

We introduce the concept of countably C-approximating posets and present characterizations and properties of countably C-approximating posets. We prove that the lattice of all σ-Scott-closed subsets of a poset is a countably C-approximating lattice, and that a complete lattice is completely distributive if and only if it is countably approximating and countably C-approximating.

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Competing Interests

The authors declare that no competing interests exist.

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