



## Covering of Line Graph of Zero Divisor Graph over Ring $\mathbb{Z}_n$

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## Abstract

Let  $\mathbb{Z}_n$  be the commutative ring of residue classes modulo  $n$ ,  $\Gamma(\mathbb{Z}_n)$  the zero divisor graph of  $\mathbb{Z}_n$  and  $L(\Gamma(\mathbb{Z}_n))$  be the line graph of  $\Gamma(\mathbb{Z}_n)$ . We have studied the point covering number and independence number of  $L(\Gamma(\mathbb{Z}_n))$ , for some positive integer  $n$ . We have computed edge covering number for  $L(\Gamma(\mathbb{Z}_{pq}))$ , and establish the relation among point covering, independence number and edge covering number of  $L(\Gamma(\mathbb{Z}_{pq}))$ , where  $p$  and  $q$  are prime numbers.

**Keywords:** Commutative Ring; Zero Divisor Graph; Line Graph; Point Covering; Independence Number; Edge Covering.

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## 1 Introduction

Let  $R$  be a commutative ring and  $Z(R)$  be its set of zero-divisors. We associate a graph  $\Gamma(R)$  to  $R$  with vertices  $Z(R)^* = Z(R) - \{0\}$ , the set of non-zero zero-divisors of  $R$  and for distinct  $u, v \in Z(R)^*$ ,

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the vertices  $u$  and  $v$  are adjacent if and only if  $uv = 0$ . In this paper, the commutative ring  $R$  is  $\mathbb{Z}_n$  and zero divisor graph  $\Gamma(R)$  is  $\Gamma(\mathbb{Z}_n)$ . The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [1], where he was mainly interested in colorings. Now, these days so many mathematicians have been working on it as C. I. Aponte, P. S. Johnson and N. A. Mims [2], H. J. Chiang-Hsieh, P. F. lee and H. J. Wang [3], E. Emad AbdAlJawad and H. Al-Ezeh [4], A. Lj. Eric and Z. S. Pucanovic [5], M. Ghanem and K. Nazzal [6].

The line graph  $L(\Gamma(\mathbb{Z}_n))$  of the  $\Gamma(\mathbb{Z}_n)$  is defined to the graph whose set of vertices constitutes of the edges of  $\Gamma(\mathbb{Z}_n)$ , where two vertices are adjacent if the corresponding edges have a common vertex in  $\Gamma(\mathbb{Z}_n)$ . The importance of line graphs stems from the fact that the line graph transforms the adjacency relations on edges to adjacency relations on vertices. For example, the chromatic index of a graph leads to the chromatic number of its line graph. We illustrate an example of a zero divisor graph and its line graph, Figure 1 shows  $\Gamma(\mathbb{Z}_{10})$  and Figure 2 shows  $L(\Gamma(\mathbb{Z}_{10}))$ .

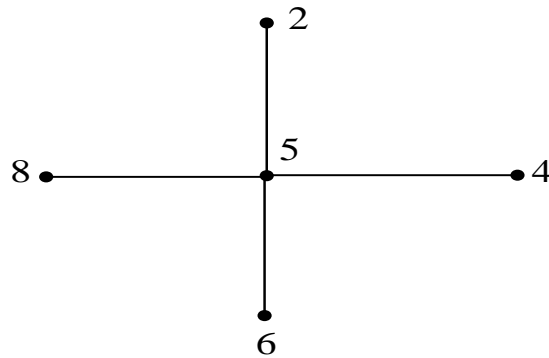


Figure 1:  $\Gamma(\mathbb{Z}_{10})$

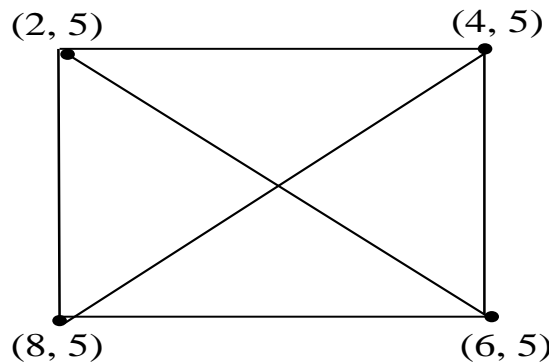


Figure 2:  $L(\Gamma(\mathbb{Z}_{10}))$

Here, we study some properties such as point covering, independence point covering and edge covering of  $L(\Gamma(\mathbb{Z}_n))$ . Also, we determine relation between point covering and edge covering.

Moreover, it is noted that if  $n$  is prime, then zero is the only zero divisor of  $\mathbb{Z}_n$ . Thus,  $\Gamma(\mathbb{Z}_n)$  is an empty graph. Also,  $\Gamma(\mathbb{Z}_4)$  has single vertex because 2 is the only non-zero zero divisor. In these cases  $L(\Gamma(\mathbb{Z}_n))$  is an empty graph which will not be studied in the sequel.

**Definition 1.1. Point covering set:-** Two points of a graph are covering each other if they are adjacent. A subset  $S$  of vertex set  $V(\Gamma(\mathbb{Z}_n))$  of vertices in a graph  $L(\Gamma(\mathbb{Z}_n))$  is a point covering set if every vertex  $v \in V(L(\Gamma(\mathbb{Z}_n)))$  is an element of  $S$  or adjacent to an element of  $S$ . The point covering number of graph is denoted by  $\alpha_{00}(L(\Gamma(\mathbb{Z}_n)))$ , defined as the cardinality of minimum point covering set of  $L(\Gamma(\mathbb{Z}_n))$ .

**Definition 1.2. Independent set:-** An independent set in a graph is a subset of vertex set  $V(L(\Gamma(\mathbb{Z}_n)))$  of  $L(\Gamma(\mathbb{Z}_n))$  such that no two vertices of subset are adjacent. The independence number of  $L(\Gamma(\mathbb{Z}_n))$ , denoted by  $\alpha'_{00}(L(\Gamma(\mathbb{Z}_n)))$ , is defined as the cardinality of a minimum independent set of  $L(\Gamma(\mathbb{Z}_n))$ .

**Definition 1.3. Minimum Edge cover:-** An edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. A minimum edge cover is an edge cover having the smallest possible number of edges for a given graph. The size of a minimum edge cover of the graph  $L(\Gamma(\mathbb{Z}_n))$  is known as the edge cover number of graph  $L(\Gamma(\mathbb{Z}_n))$  and is denoted by  $\rho(L(\Gamma(\mathbb{Z}_n)))$ .

## 2 Results

**Theorem 2.1.** *Let  $L(\Gamma(\mathbb{Z}_n))$  be a line graph of  $\Gamma(\mathbb{Z}_n)$ , where  $n = 2p$ , and  $p$  is an odd prime number. Then point covering and independence number of  $L(\Gamma(\mathbb{Z}_n))$ , both are one.*

*Proof.* When  $n = 2p$ , then  $\Gamma(\mathbb{Z}_n)$  is a star graph. So there is a common vertex which is adjacent to all other vertices and that vertex is also called center of the graph. When we draw the line graph of  $\Gamma(\mathbb{Z}_n)$ , for  $n = 2p$ , and let  $v_1$  be the common vertex of  $\Gamma(\mathbb{Z}_n)$  which is the end point of every edge of  $\Gamma(\mathbb{Z}_n)$ . Then  $v_1$  appears in every vertex of the line graph.  $[v_1, u_i] \in V(L(\Gamma(\mathbb{Z}_n)))$ , where  $\{i = 2, 2.2, \dots, 2(p-1), p = v_1\}$  forms a complete line graph of  $\Gamma(\mathbb{Z}_n)$  and here,  $[v_1, u_1]$  is adjacent with all other vertices of line graph. In other words, we can say that single vertex cover all other vertices of line graph of zero divisor graph  $\Gamma(\mathbb{Z}_n)$ . Thus, point cover is one and that vertex is also use for independence number. □

**Theorem 2.2.** *For  $L(\Gamma(\mathbb{Z}_n))$ , if  $n = 3p$ , where  $p$  is an odd prime number, then point covering number and independence number are two.*

*Proof.*  $\Gamma(\mathbb{Z}_n)$  is a complete bi-partite graph, when  $n = 3p$ . Then, there are two independent set of vertices, one set has  $p-1$  elements which are multiple of 3 and second set has  $(3-1)$  elements multiple of  $p$ . In the zero divisor graph  $\Gamma(\mathbb{Z}_n)$ , there are two vertices which are adjacent to all other vertices but not to each other. These two vertices of  $\Gamma(\mathbb{Z}_n)$  appears in line graph as the end points of edges. Let  $[u_1, v_1]$  and  $[u_2, v_2] \in V(L(\Gamma(\mathbb{Z}_n)))$ . Then they are not adjacent to each other but  $[u_1, v_1]$  adjacent to  $[u_1, v_i] \in V(L(\Gamma(\mathbb{Z}_n)))$ , where  $v_i$  is multiple of 3. Similarly,  $[u_2, v_2]$  adjacent to  $[u_2, v_i] \in V(L(\Gamma(\mathbb{Z}_n)))$ . Then  $[u_1, v_1]$  and  $[u_2, v_2]$  are two vertices covers all vertices of line graph of  $\Gamma(\mathbb{Z}_n)$ . So, independence number is two. Again,  $[u_1, v_1]$  and  $[u_2, v_1]$  cover all vertices of line graph of  $\Gamma(\mathbb{Z}_n)$ , which are adjacent to each other. Thus, point covering is also two. □

**Theorem 2.3.** *If  $n = pq$ ,  $p, q$  are odd prime number and  $p < q$ , then line graph of  $\Gamma(\mathbb{Z}_n)$  has  $p - 1$  point covering number and  $p - 1$  independence number.*

*Proof.* For  $n = pq$ ,  $\Gamma(\mathbb{Z}_n)$  is a complete bi-partite graph. So, there are two independent sets of vertices in  $\Gamma(\mathbb{Z}_n)$ . One independent set has  $p - 1$  number of elements which are multiple of  $q$  and adjacent to all vertices but not adjacent with each other. When we consider line graph of  $\Gamma(\mathbb{Z}_n)$ , then  $p - 1$  vertices appearing in line graph as the end points of edges. Let  $[u_1, v_1]$  and  $[u_2, v_2] \in V(L(\Gamma(\mathbb{Z}_n)))$ . Then, they are not adjacent to each other but  $[u_1, v_1]$  adjacent to  $[u_1, v_i] \in V(L(\Gamma(\mathbb{Z}_n)))$ , where  $v_i$  is multiple of  $p$ .  $[u_2, v_2]$  adjacent with  $[u_2, v_i]$  and  $[u_j, v_2]$ , where  $v_i$  is multiple of  $p$  and  $u_j$  is multiple of  $q$ . Similarly, the process is continuous upto  $j = 1, 2, \dots, (p - 1)$ . Thus,  $[u_1, v_1], [u_2, v_2], \dots, [u_{p-1}, v_i]$  cover all vertices but not adjacent to each other. They are independent vertices and  $p - 1$  in number. Therefore, independence number is  $p - 1$ . On the other hand,  $[u_1, v_1], [u_2, v_1], \dots, [u_{p-1}, v_1]$  vertices cover all vertices and they are adjacent to each other. Hence, point covering is  $p - 1$ . □

**Theorem 2.4.** *Point covering and independence numbers are  $\frac{p-1}{2}$  in line graph  $L(\Gamma(\mathbb{Z}_n))$ , where  $n = p^2$ , for  $p$  is any prime number.*

*Proof.* If  $n = p^2$ , then  $\Gamma(\mathbb{Z}_n)$  is  $K_{p-1}$  complete graph for every prime  $p$ . Then, there are  $p - 1$  vertices which are adjacent to each other. When we draw the line graph of  $\Gamma(\mathbb{Z}_n)$ , then it is a regular graph with  $(\frac{\sum deg(p-1)}{2})$  vertices and each vertex  $[v_1, u_1]$  is adjacent with  $(2p-6)$  vertices, where  $v_1$  cover  $(p-3)$  vertices and  $u_1$  cover next  $p-3$  vertices. Therefore,  $[v_1, u_1]$  covers  $(2p-6)$  vertices. Hence, graph is regular and each vertex of line graph covers  $(2p-6)$  vertices. We know that each vertex of line graph is an end point of each edge of zero divisor graph of  $\mathbb{Z}_n$ . Therefore,  $\frac{p-1}{2}$  vertices cover all other vertices and they are adjacent to each other. Hence point covering of  $L(\Gamma(\mathbb{Z}_n))$  is  $\frac{p-1}{2}$ . There are  $\frac{p-1}{2}$  points which are not adjacent to each other but cover all vertices. Therefore, independence number is  $\frac{p-1}{2}$ . □

**For Example:-** 1. When  $p = 2$ , there is no vertex in  $L(\Gamma(\mathbb{Z}_4))$ . Hence, point covering and independence number are zero.

2. When  $p = 3$ , then there is a single vertex in  $L(\Gamma(\mathbb{Z}_9))$ , and in this case covering point and independence number are one.

3. When  $p = 5$ , there are six vertices in  $L(\Gamma(\mathbb{Z}_{25}))$ , then point covering is two and independence number is also two.

**Theorem 2.5.** *If  $n = p^3$ , then point covering and independence number of  $L(\Gamma(\mathbb{Z}_n))$  is one, where  $p = 2$  or  $3$ .*

*Proof.* In zero divisor graph of  $\mathbb{Z}_n$ , where  $n = p^3$ ,  $\Gamma(\mathbb{Z}_n)$  is complete 3-partite graph. There are  $p - 1$  vertices which are adjacent to each other and adjacent with all multiples of  $p$ . These  $(p - 1)$  elements has minimum eccentricity and form centers of  $\Gamma(\mathbb{Z}_n)$ . But in the line graph of  $\Gamma(\mathbb{Z}_n)$ ,  $\frac{p-1}{2}$  elements are centers. These  $\frac{p-1}{2}$  elements are adjacent to each other and also adjacent with multiple elements of  $p$ . Therefore, a single vertex which is also a center, that cover all vertices of line

graph and minimum in numbers. Thus, point covering is one and the same vertex which covers all vertices is also give independence number. □

**Theorem 2.6.** *If  $n = p^3$ , then point covering and independence number of  $L(\Gamma(\mathbb{Z}_n))$  are  $\frac{p-1}{2}$ , where  $p > 2$  or  $p > 3$ .*

**For Example:-** For  $L(\Gamma(\mathbb{Z}_n))$ :

1. When  $n = 8$ , the center is  $[2, 4]$  and  $\varepsilon(v) = 1$ , for every  $v \in L(\Gamma(\mathbb{Z}_8))$ , the center covers all the vertices.



Figure 3:  $L(\Gamma(\mathbb{Z}_8))$

2. When  $n = 27$ , the center is  $[9, 18]$  and  $\varepsilon[9, 18] = 1$ , for every vertex of line graph of  $\Gamma(\mathbb{Z}_{27})$ , which cover all the vertices.

3. When  $n = 125$ , the centers are  $[25, 50], [25, 75], [25, 100], [50, 75], [50, 100], [75, 100]$  and each one has eccentricity two. So we can take two elements from centers which covers all the vertices.

Hence, from above two examples, when  $n = p^3$ , where  $p = 2$  or  $3$ , point covering and independence number are one. When  $p \neq 2$  and  $p \neq 3$ , then point covering and independence number are  $\frac{p-1}{2}$ .

**Theorem 2.7.** *Let  $p$  be a prime number and  $n = 2p$ , for  $p > 2$  and  $n = 3p$ , for  $p > 3$ . Then edge covering of  $L(\Gamma(\mathbb{Z}_n))$  is one.*

*Proof. Case 1:-* When  $n = 2p$ , the line graph of  $\Gamma(\mathbb{Z}_n)$  is a complete graph. Therefore, every vertex is adjacent to each other. Let  $[v_1, u_1]$  and  $[v_1, u_2] \in V(L(\Gamma(\mathbb{Z}_n)))$ . Then there is an edge between  $[v_1, u_1]$  and  $[v_1, u_2]$  and these are end points of that edge. There is  $v_1$  point appears in every vertex in line graph of  $\Gamma(\mathbb{Z}_n)$ . Since we know for  $n = 2p$ ,  $\Gamma(\mathbb{Z}_n)$  is a star graph, so let  $v_1$  be the center of  $\Gamma(\mathbb{Z}_n)$  which covers all vertices. Therefore, edge of vertices  $[v_1, u_1]$  and  $[v_1, u_2]$  covers all the vertices which is also minimum in number. Hence, edge covering is one.

**Case 2:-** If  $n = 3p$ , then  $\Gamma(\mathbb{Z}_n)$  is a complete bipartite graph and line graph of  $\Gamma(\mathbb{Z}_n)$  is a regular graph. If  $[v_1, u_1]$  and  $[v_1, u_2] \in V(L(\Gamma(\mathbb{Z}_n)))$ , then there is an edge between them. We know that

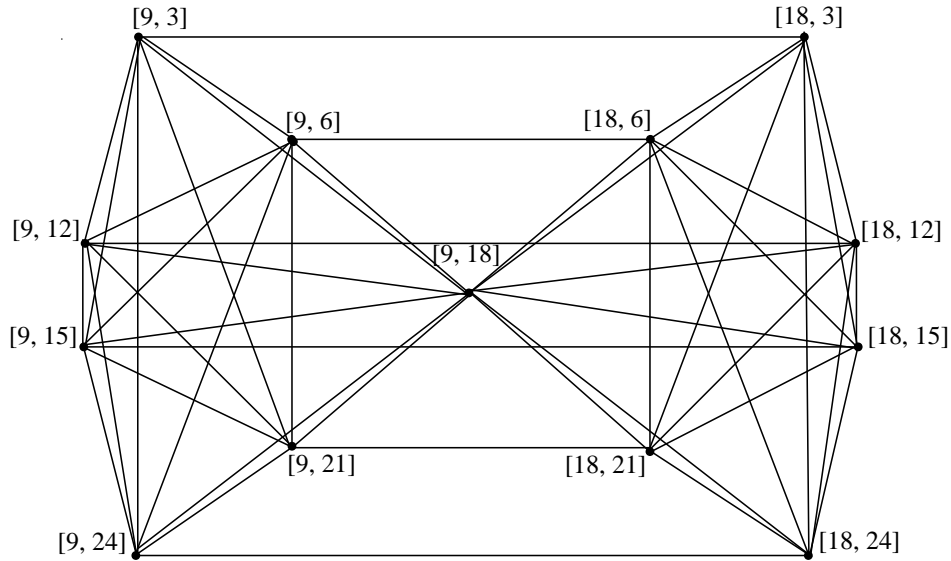


Figure 4:  $L(\Gamma(\mathbb{Z}_{27}))$

when  $n = 3p$ ,  $\Gamma(\mathbb{Z}_n)$  has  $(3 - 1)$  points, which are multiples of  $p$  and covers all vertices. Let  $u_1, u_2$  be multiples of  $p$  and they are two end points of an edge  $[v_1, u_1], [v_1, u_2]$  in line graph  $L(\Gamma(\mathbb{Z}_n))$ . Thus, one edge covers all the vertices. Therefore, edge covering is one. □

**Theorem 2.8.** Edge covering in  $L(\Gamma(\mathbb{Z}_n))$  is  $\lceil \frac{p-1}{2} \rceil$ , where  $n = pq$ ,  $p < q$  and  $p, q$  are prime numbers.

*Proof.* If  $n = pq$ , then zero divisor graph  $\Gamma(\mathbb{Z}_n)$  is a complete bipartite graph. So there are two independent set of vertices, in which every vertex of one set is adjacent to every vertices of other set. If  $p < q$ , then  $\Gamma(\mathbb{Z}_n)$  has  $(p - 1)$  vertices which are multiples of  $q$  and covers all vertices of  $\Gamma(\mathbb{Z}_n)$ . We draw the line graph of  $\Gamma(\mathbb{Z}_n)$  and suppose  $[v_1, u_1]$  and  $[v_2, u_1]$  are two vertices of line graph of  $\Gamma(\mathbb{Z}_n)$ . Then, there exists an edge between them. One end  $[v_1, u_1]$  of the edge covers  $[v_1, u_i]$  and other end  $[v_2, u_1]$  covers  $[v_2, u_i]$  vertices of  $L(\Gamma(\mathbb{Z}_n))$ , where  $i = p, 2p, \dots, (q - 1)p$ . Again, we take another edge  $[v_3, u_2], [v_4, u_2]$  are also cover vertices  $[v_3, u_i]$  and  $[v_4, u_i]$  respectively, where  $i = p, 2p, \dots, (q - 1)p$ . The process is going on till all vertices are not covered. There are  $\frac{p-1}{2}$  edges that covers all vertices of  $L(\Gamma(\mathbb{Z}_n))$ , i.e.  $([u_1, v_1], [u_2, v_1], \dots, [u_{p-1}, v_1])$  are vertices cover all vertices and adjacent to each other and they are end points of  $\frac{p-1}{2}$  edges. and they are minimum in numbers. Thus, edge covering of  $L(\Gamma(\mathbb{Z}_n))$  is  $\lceil \frac{p-1}{2} \rceil$ , where  $n = pq$ ,  $p < q$  and  $p$  and  $q$  are prime numbers. □

**Theorem 2.9.** Edge covering is less than the point covering and independence number in  $L(\Gamma(\mathbb{Z}_n))$ , where  $n = pq$ ,  $p > 2$ ,  $p$  and  $q$  are distinct prime numbers.

*Proof.* If  $n = 2p$ , then point covering and edge covering is one. (From Theorem (1) and Theorem (6) (case-1))

If  $n = pq$ , and  $p > 2$ , then from Theorem (2) to Theorem (7), shown that edge covering is always less than to point covering and independence number.

It is also easily seen that point covering and independence numbers are equal. □

### 3 Conclusions

In this paper, we studied the point covering number and independence number associated with line graph of zero divisor graph over commutative ring  $\mathbb{Z}_n$ . Some details are given below.

- a In the section (1), a brief historical background and definitions of line graph related to  $\mathbb{Z}_n$  has been discussed. Also, importance of line graph is given.
- b In the section (2), we have studied the point covering number and independence number of line graph of  $\mathbb{Z}_n$ , where  $n$  is product of primes. Also, we discussed the edge covering number of line graph over  $\mathbb{Z}_{pq}$ . Furthermore, we have established the relation among point covering number, independence number and edge covering number and concluded that edge covering number is less than the point covering number and independence number in  $L(\Gamma(\mathbb{Z}_{pq}))$  where  $n = pq$ .

### Competing Interests

The authors declare that no competing interests exist.

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