



Using Eight-Variable Karnaugh Maps to Unravel Hidden Technicalities in Qualitative Comparative Analysis

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Authors' contributions

This work was carried out in collaboration between both authors. Author AMAR designed the study, performed the analysis, solved the example, deliberated over the QCA paradigm, contributed to the literature search, and wrote the preliminary manuscript. Author RSB managed the literature search, deliberated over the QCA paradigm, and drew the figures. Both authors read and approved the final manuscript.

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ABSTRACT

We use a regular and modular eight-variable Karnaugh map to reveal some technical details of Boolean minimization usually employed in solving problems of Qualitative Comparative Analysis (QCA). We utilize as a large running example a prominent eight-variable political-science problem of sparse diversity (involving a partially-defined Boolean function (PDBF), that is dominantly unspecified). We recover the published solution of this problem, showing that it is merely one candidate solution among a set of many equally-likely competitive solutions. We immediately obtain one of these rival solutions, that looks better than the published solution in two aspects, namely: (a) it is based on a smaller minimal set of supporting variables, and (b) it provides a more compact Boolean formula. However, we refrain from labelling our solution as a better one, but instead we stress that it is simply un-comparable with the published solution. The comparison between any two rival solutions should be context-specific and not tool-specific. In fact, the

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Boolean minimization technique, borrowed from the area of digital design, cannot be used as is in QCA context. A more suitable paradigm for QCA problems is to identify all minimal sets of supporting variables (possibly via integer programming), and then obtain all irredundant disjunctive forms (IDFs) for each of these sets. Such a paradigm stresses inherent ambiguity, and does not seem appealing as the QCA one, which usually provides a decisive answer (irrespective of whether it is justified or not). The problem studied herein is shown to have at least four distinct minimal sets of supporting variables with various cardinalities. Each of the corresponding functions does not have any non-essential prime implicants, and hence each enjoys the desirable feature of having a single IDF that is both a unique minimal sum and the complete sum. Moreover, each of them isunate as it is expressible in terms of un-complemented literals only. Political scientists are invited to investigate the meanings of the (so far) abstract formulas we obtained, and to devise some context-specific tool to assess and compare them.

Keywords: Criticism of QCA; prime implicant; irredundant disjunctive form; minimal sum; complete sum; partially-defined boolean function; set of supporting variables.

1. INTRODUCTION

Social sciences, striving to undergo the mathematization that pure sciences earlier went through, are currently making use of more powerful and involved mathematical insight and techniques [1]. One of these is the emergent methodology of Qualitative Comparative Analysis (QCA), which imports techniques of Boolean Analysis from Electrical and Computer Engineering [2-8]. Thiem and Duşa [3] assert that the success of QCA as a social research method is attested to by (a) an increase in the number of its empirical applications, and (b) the widening landscape of computer programs tailored to serve it. Despite this, QCA has been the subject of many kinds of criticism [6,8-17]. This paper is yet another attempt to scrutinize and constructively criticize QCA.

Rushdi and his associates [18-19] conjecture that QCA would be trusted only if it produces a Boolean function whose prime implicants are all essential [20-24]. This happens when the function possesses only a single irredundant disjunctive form (IDF) that is both a unique minimal sum and the complete sum (the Blake canonical form [20, 25]. An example of such a function is a unate Boolean function, which is a function in which each variable is mono-form (either complemented or non-complemented) [26-29]. The above conjecture reconciles the aspiration for minimality by mainstream QCA researchers, the search for IDFs by Crama et al. [30] and Thiem et al. [8], and the insistence on including the complete sum by Rushdi [15] and Rushdi & Badawi [6]. The search for a coherent Boolean function in a QCA solution by Rushdi & Rushdi [16] conforms to this conjecture. The running example considered herein (taken from Dumont and Bäck [31], and the

one in [19] (taken from Delreux [32] are also in favor of the validity of this conjecture.

The main thesis of this paper is that the Boolean minimization technique, borrowed from the area of digital design, cannot be used as is in QCA context. A more suitable paradigm for QCA problems is to identify all minimal sets of supporting variables (possibly via integer programming, as in [30], and then obtain all irredundant disjunctive forms (IDFs) for each of these sets. Such a paradigm stresses inherent ambiguity, and does not seem appealing as the QCA one, which usually provides a decisive answer (irrespective of whether it is justified or not). This paper follows [6,12,13,15,19] in arguing that the Boolean Analysis necessitated by QCA is preferably achieved by the manual pictorial tool of the Karnaugh map rather than by automated algorithms. In [19], we lamented that QCA implementation of the Quine-McCluskey algorithm (QMC) [33-35] via computer programs is a fast way of obtaining hasty results without questioning or scrutinizing the underlying recipes, which might be black boxes for the users. Admittedly, the Karnaugh map is useful only for a small number of input variables. Nevertheless, the map plays an indispensable role in logic design, although typical real-life problems therein are large (and even extremely large). The size-limitation of the Karnaugh map, however, is usually of no grave concern in the case of Boolean Analysis needed for QCA, where the typical number of input variables of 'real-life' problems seems to be eight to ten at most [3,19,31,32,36]. Some limited use of Karnaugh maps (of four or fewer variables) has been already made in QCA applications [3,12,13,15,16,37,38].

In this paper, we introduce a novel regular and modular eight-variable Karnaugh map, first introduced by Halder [39] and Motil [40], and efficiently used for the study of multi-state reliability [41] and for an exposition of four-bit magnitude digital comparators [18,42]. We employ this map for analyzing one of the largest QCA problems ever studied [31]. We compress the eight input binary variables as two hexadecimal variables, so as to facilitate the conversion of the original truth table into an eight-variable Karnaugh map. Only observed configurations are explicitly shown in map cells entered as asserted high (1) or asserted low (0). Non-observed configurations corresponding to logical remainders (don't cares, in our language) are represented with blank cells (as usual).

The eight-variable problem studied herein is shown to have at least four distinct minimal sets of supporting variables with various cardinalities, including the set implied by the solution in [31]. Each of the corresponding functions studied does not have any non-essential prime implicant, and hence each of them enjoys the desirable feature of having a single IDF that is both a unique minimal sum and the complete sum. Moreover, each of them is unate and expressible in terms of un-complemented literals only. Political scientists are invited to investigate the meanings of the (so far) abstract formulas we obtained, and to devise some context-specific tool to assess and compare them.

The organization of the rest of this paper is as follows. Section 2 introduces the regular and modular version of the Karnaugh map used in this paper. This map version can be (theoretically) extended to an arbitrary large number of variables, and includes all maps of smaller sizes as special cases. Section 3 presents the QCA problem of Dumont and Bäck [31] used as a running example herein. Section 4 revisits this problem from a map perspective, first by using the classical map procedure, and later by insisting on identifying a minimal set of supporting variables before proceeding to obtain IDF solutions. Subsequently, Section 4 contributes three rival (improved?) solutions. Section 5 concludes the paper.

2. ON THE REGULAR EIGHT-VARIABLE KARNAUGH MAP

The Karnaugh map is an enhanced form of the truth table [19,43] in which, (a) two dimensions (rather than one) are used, and (b) reflected

binary ordering or grey ordering (as opposed to direct binary ordering) is employed. The n -variable map consists of 2^n cells, such that every cell has n neighboring cells or logically adjacent cells. Logical adjacency pertains to the concept of the Hamming distance between the bit strings representing the cells, which is the number of bit positions in which the two bits are different [44-49]. Two cells are (first) neighbors or (immediately) adjacent if their variable values except one are exactly the same. The map is constructed such that any two logically adjacent cells are made also as visually adjacent as possible. In general, two logically adjacent cells appear as the mirror images with respect to boundary lines separating the internal and external domains of the variable in whose value the two cells differ. Fig. 1 shows the regular and modular form of the Karnaugh map to be used herein, which is such that

- a) The rectangular shape of the cell is retained.
- b) The internal domain of the $(n + 1)$ st variable is introduced to be centered around the boundary lines of the $(n - 1)$ st variable.

We note that the process outlined in (b) above can be, in theory, indefinitely continued. Hence, there is no theoretical upper bound on the size of the Karnaugh map constructed this way. However, as the number of variables increases, the size of the map increases exponentially, and its utility diminishes very quickly due to prohibitively increasing difficulty. As a demonstration of the usefulness of the aforementioned version of the n -variable Karnaugh map, we present its case of eight variables in this paper. See Fig. 1. This map suffices to handle most problems of QCA.

The construction of a Karnaugh map is based on the Boole-Shannon Expansion and its intimately-related concept of a subfunction [50-51], Boolean quotient [20,52], or restriction [53-54]. This expansion takes the following form when implemented *w.r.t.* a single variable X_k

$$f(X) = (\bar{X}_k \wedge f(X|0_k)) \vee (X_k \wedge f(X|1_k)) \quad (1)$$

This Boole-Shannon Expansion expresses the Boolean function $f(X)$ in terms of its two subfunctions $f(X|0_k)$ and $f(X|1_k)$. These subfunctions are equal to the Boolean quotients $f(X)/\bar{X}_k$ and $f(X)/X_k$, and hence are obtained by restricting X_k in the expression $f(X)$ to 0 and 1

respectively. If $f(X)$ is a sum-of-products expression of n variables, the two subfunctions $f(X|0_k)$ and $f(X|1_k)$ are functions of at most $(n - 1)$ variables. If the Boole-Shannon expansion is applied in sequence to the n variables of $f(X)$, the expansion tree is a complete binary tree (usually called a Binary Decision Diagram) of 2^n leaves. These leaves are functions of no variables, or constants, and equal the entries of a corresponding conventional Karnaugh map of $f(X)$. Sibling nodes (nodes at the same level) of this expansion tree constitute the entries of a variable-entered (or a map-entered) Karnaugh map of $f(X)$ [55].

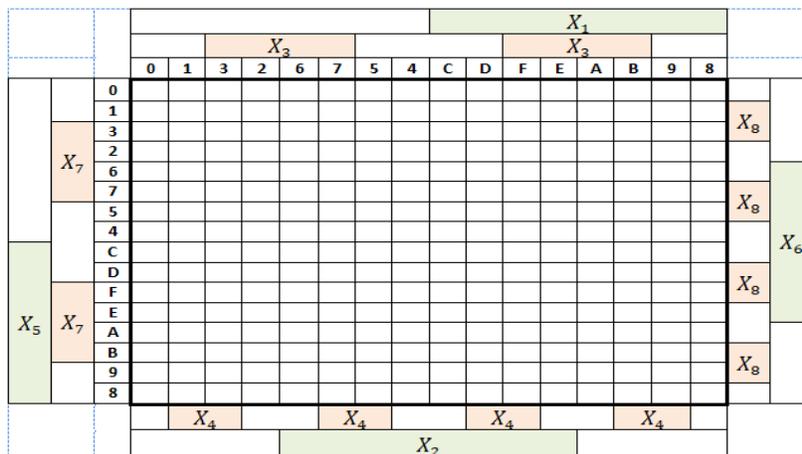
If a Karnaugh map is used to represent a Boolean function $f(X)$, then the map can be split into two halves (with respect to the borders of the variable X_k) representing the internal and external domains of this variable. These half maps depict, respectively, the two subfunctions or Boolean quotients $f(X)/\bar{X}_k$ and $f(X)/X_k$, which are functions of (at most) the $(n - 1)$ variables of X other than X_k . We say that $f(X)$ is vacuous in (independent of) X_k if the following relation is identically satisfied [12,48, 49,56-59]

$$f(X)/\bar{X}_k = f(X)/X_k \quad (2)$$

According to (1), if (2) is satisfied then $f(X)$ becomes equal to each of $f(X)/\bar{X}_k$ and $f(X)/X_k$, and hence it becomes a function of (at most) the $(n - 1)$ variables of X other than X_k . This means that X_k is now guaranteed to be not a supporting variable of $f(X)$. The variable X_k must be

included in a set of supporting variables of $f(X)$ if at least a cell in the half map $f(X)/\bar{X}_k$ is found to have an entry differing from that of its image cell in the half map $f(X)/X_k$ (obtained through reflection with respect to the nearest border of X_k). Note that only the 0 and 1 entries are viewed as different, opposing or contradictory, while a don't-care entry (d) can be made to match either 0 or 1. The variable X_k might be excluded from a set of supporting variables of $f(X)$ if (a) no case of the afore-mentioned pair of contradictory image cells can be found, and (b) appropriate conditions are placed on don't-care cells in either the $f(X)/\bar{X}_k$ or $f(X)/X_k$ half maps that are images of 0-entered or 1-entered cells in the other half so as to ensure that (2) is satisfied.

In passing, we note that Occam's razor or the principle of parsimony [60-66] is usually alluded to justify minimization strategies. This (dubious? Often misused?) principle states that 'entities should not be multiplied without necessity' or that 'the simplest explanation is usually the right one.' While streamline QCA usually takes this principle to its extreme by seeking a minimal sum of the pertinent Boolean function, the strategy of Crama et al. [30] adopted herein uses this principle cautiously by listing *all* minimal sets of supporting variables, and then listing *all* possible IDFs for each of these sets. Two issues related to this strategy concern the cardinality of a minimal set of supporting variables [67], and the best and worst IDF, i.e., the one with minimum and maximum numbers of prime implicants [68].



Z

Fig. 1. A regular and modular eight-variable Karnaugh map. For convenience, the numerical values of the hexadecimal numbers are shown for the sets of horizontal and vertical variables

3. A RUNNING EXAMPLE

In Table 1, we present the running example to be used throughout this paper, which is taken from Dumont and Bäck [31]. This example deals with the political-science question of governmental participation of green parties [69-77]. Table 1 is a truth table with a single binary endogenous factor (outcome) Z and a set of eight binary exogenous (input) factors or conditions ($X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8$). All variables of Dumont and Bäck [31] have lengthy mnemonic names conveying technical meanings Table 2, which are renamed herein with simple abstract symbols, so as to facilitate the reconstruction of the truth table as a Karnaugh map. The set of eight input factors or variables is split into two distinct sets of variables $X_H = (X_1, X_2, X_3, X_4)$ and $X_L = (X_5, X_6, X_7, X_8)$, which serve as the horizontal and vertical variables of the Karnaugh map, respectively. Each row in Table 1 represents a unique configuration that is characterized by a binary number $(X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8)_2$ or an equal hexadecimal number $(X_H X_L)_{16}$ Table 3, where

$$X_H = 2^3 X_1 + 2^2 X_2 + 2^1 X_3 + 2^0 X_4 = 8X_1 + 4X_2 + 2X_3 + X_4 \quad (3)$$

$$X_L = 2^3 X_5 + 2^2 X_6 + 2^1 X_7 + 2^0 X_8 = 8X_5 + 4X_6 + 2X_7 + X_8 \quad (4)$$

Note that the hexadecimal number system uses 16 symbols, the first 10 of which (0,1,2,...,9) are borrowed (with the same meaning) from the conventional decimal system, while the remaining 6 symbols are the beginning uppercase letters of the alphabet A, B, C, D, E, and F used to designate the values 10, 11, 12, 13, 14, and 15, respectively, in decimal, which correspond to 1010,1011,1100,1101,1110 and 1111, respectively in binary Table 3. Designation of a configuration in Table 1 by a hexadecimal number $X_H X_L$ considerably facilitates the conversion of the truth table in Table 1 to the Karnaugh map in Fig. 2 or Fig. 3. The configuration $X_H X_L$ is simply located at the horizontal coordinate X_H and the vertical coordinate X_L . The output variable Z is a binary variable of a value belonging to $\{0,1\}$, to be entered in the Karnaugh map.

The truth table in Table 1 is remarkable for its extremely few specifications. Out of $2^8 = 256$ possible configurations, data is available for only 36 configurations. The fact is vividly illustrated by Fig. 3, in which the Karnaugh map has 256 cells, with only 36 of them of assigned asserted entries of 0 or 1. The remaining 220 cells are left blank,

indicating that they are logical remainders (in QCA terminology) or don't-cares (in our digital-design terminology).

Dumont and Bäck [31] obtained the following solution for their problem

$$Z = X_1 X_2 (X_6 \vee X_5 X_8) = X_1 X_2 X_6 \vee X_1 X_2 X_5 X_8 \quad (5)$$

This solution assumes a set of supporting variables $\{X_1, X_2, X_5, X_6, X_8\}$ of cardinality five, or it discovers this set as an afterthought. We have an obvious objection on the uniqueness of this supporting set of variables and the corresponding solution of Z . In the next section, we will recover this solution, but we will find that it is not a unique solution, but it is one out of several rival solutions that are based on different supporting sets of variables. For more information on the concept of sets of supporting variables of a Boolean function, the interested reader might refer to [30,78-85].

Our afore-mentioned objection is not pertaining to the result in [31] only, but is expected to arise frequently against many other QCA solutions. Despite our repeated criticism of the way of handling logical remainders in QCA circles [6, 12, 13, 15, 16, 19], no adequate response from the QCA community is obtained so far, and the mainstream QCA journals have never addressed the issue at all, as if it were of no genuine scientific concern. It would have been in the interest of science and truth, if these journals had tried to carefully listen to outsiders and to actively consider other people's points of view.

4. MAP DERIVATION OF SUPPORTING SETS

Fig. 1 shows the regular and modular eight-variable Karnaugh map to be used herein. For convenience, the numerical values of the hexadecimal numbers are shown for the sets of horizontal and vertical variables $X_H = (X_1, X_2, X_3, X_4)$ and $X_L = (X_5, X_6, X_7, X_8)$. Fig. 2 shows locations of specified configurations of the running example as shown in Table 1 (taken from Dumont and Bäck [31]). The locations are depicted as two hexadecimal numbers, and colored green for high assertion (1) and red for low assertion (0). Our running example is henceforth represented by the 8-variable Karnaugh map in Fig. 3, in which the location entries are replaced by the value entries of 1 for high assertion and 0 for low assertion. Fig. 4 is a Karnaugh-map demonstration of the solution of

the running example that was obtained via automated means by Dumont and Bäck [31]. This solution assumes a set of supporting variables $\{X_1, X_2, X_5, X_6, X_8\}$ of cardinality five, and is given by Equation (5). In the sequel, we show that this solution is neither unique nor best (in QCA sense). Fig. 5 shows another solution of the running example obtained via direct map inspection. This solution assumes a set of supporting variables $\{X_1, X_2, X_4, X_6\}$ of cardinality four only, and is given by

$$Z = X_1 X_2 (X_6 \vee X_4) = X_1 X_2 X_6 \vee X_1 X_2 X_4 \quad (6)$$

We do not claim that this solution is better than the one in (5), though it is more compact than it and uses fewer supporting variables than it. We believe that both (5) and (6) are valid candidate solutions, whose rivalry could be settled only on context-specific grounds. Since we are not political scientists, we refrain from attempting to judge which of these two (and other possible) solutions makes better sense. We are not going to exhaust all possible solutions herein, but we will give ample details for deriving two more solutions.

Table 1. Truth table of the running example, adapted from Table 3 in dumont and bäck [31]. Only 36 out of 256 lines or configurations are specified. A line 01100-00/0 is split here into two lines

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_H	X_L	Z
0	1	1	0	0	0	0	0	6	0	0
0	1	1	0	0	0	0	1	6	1	0
0	1	1	0	0	0	1	1	6	3	0
1	1	1	0	0	0	1	0	E	2	0
1	1	0	0	0	0	1	1	C	3	0
1	0	0	0	0	0	1	1	8	3	0
0	1	0	0	1	0	0	0	4	8	0
0	0	0	0	1	0	0	0	0	8	0
0	0	1	1	1	0	1	0	3	A	0
0	1	1	0	1	0	1	1	6	B	0
1	0	1	0	1	0	1	1	A	B	0
1	1	1	0	1	0	1	0	E	A	0
0	0	1	1	0	1	0	0	3	4	0
0	1	1	1	0	0	0	0	7	0	0
0	1	1	1	1	0	1	0	7	A	0
0	1	0	1	1	0	1	1	5	B	0
0	1	0	0	1	0	1	1	4	B	0
1	0	1	1	1	0	1	1	B	B	0
0	1	0	0	1	0	0	1	4	9	0
0	0	0	0	1	0	1	0	0	A	0
1	0	0	0	1	0	1	1	8	B	0
0	0	1	1	1	0	0	0	3	8	0
0	0	1	1	0	0	0	0	3	0	0
0	1	1	0	0	0	0	0	6	0	0
0	1	1	0	0	1	0	0	6	4	0
0	0	1	0	0	0	0	0	2	0	0
0	1	1	0	1	0	0	0	6	8	0
1	0	1	0	0	0	0	0	A	0	0
1	0	1	1	0	0	1	1	B	3	0
0	1	1	0	1	0	0	1	6	9	0
1	0	1	0	1	1	0	1	A	D	0
1	0	1	0	1	1	1	0	A	E	0
1	1	1	0	1	1	1	1	E	F	1
1	1	1	1	1	1	1	0	F	E	1
1	1	1	1	1	1	1	1	F	F	1
1	1	1	1	1	0	1	1	F	B	1

		X ₃								X ₃							
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8
X ₅	X ₇	0		30	20	60	70							A0			
		1				61											
		3				63				C3					B3		83
		2											E2				
		6															
	X ₇	7															
		5															
		4		34			64										
		C															
		D													AD		
X ₇	F										FF	EF					
	E										FE		AE				
	A	0A		3A			7A						EA				
	B					6B		5B	4B		FB		AB	BB		8B	
	9					69			49								
8	08		38			68			48								
		X ₄				X ₄				X ₄				X ₄			
		X ₂															

Z

Fig. 2. Locations of specified configurations of the running example (taken from Dumont and Bäck [31], depicted as two hexadecimal numbers, and colored green for high assertion (1) and red for low assertion (0). For convenience, the numerical values of the hexadecimal numbers are shown

		X ₃								X ₃							
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8
X ₅	X ₇	0		0	0	0	0							0			
		1				0											
		3				0				0					0		0
		2											0				
		6															
	X ₇	7															
		5															
		4		0			0										
		C															
		D													0		
X ₇	F										1	1					
	E										1		0				
	A	0		0			0					0					
	B					0		0	0		1		0	0		0	
	9					0			0								
8	0		0					0									
		X ₄				X ₄				X ₄				X ₄			
		X ₂															

Z

Fig. 3. Specifications of the cells of high assertion (1) and low assertion (0) for the running example, taken from Dumont and Bäck [31]

												X ₁					
		X ₃										X ₃					
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8
X ₅	X ₇	0		0	0	0	0							0			
		1				0											
		3				0				0					0		0
		2											0				
		6															
	X ₇	7															
		5															
		4		0		0											
		C															
		D													0		
X ₇	F										1	1					
	E										1			0			
	A	0		0			0						0				
	B					0		0	0		1			0	0	0	
	9					0			0								
8	0		0		0			0									
		X ₄					X ₄					X ₄				X ₄	
		X ₂															

$$\begin{aligned}
 Z &= X_1 X_2 (X_6 \vee X_5 X_8) \\
 &= X_1 X_2 X_6 \vee X_1 X_2 X_5 X_8
 \end{aligned}$$

Fig. 4. Solution of the running example obtained via automated means by dumont and bäck [31]. This solution assumes a set of supporting variables {X₁, X₂, X₅, X₆, X₈} of cardinality five

												X ₁					
		X ₃										X ₃					
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8
X ₅	X ₇	0		0	0	0	0							0			
		1				0											
		3				0				0					0		0
		2											0				
		6															
	X ₇	7															
		5															
		4		0		0											
		C															
		D													0		
X ₇	F										1	1					
	E										1			0			
	A	0		0			0						0				
	B					0		0	0		1			0	0	0	
	9					0			0								
8	0		0		0			0									
		X ₄					X ₄					X ₄				X ₄	
		X ₂															

$$\begin{aligned}
 Z &= X_1 X_2 (X_6 \vee X_4) \\
 &= X_1 X_2 X_6 \vee X_1 X_2 X_4
 \end{aligned}$$

Fig. 5. Another (better?) solution of the running example obtained via map inspection. This solution assumes a set of supporting variables {X₁, X₂, X₄, X₆} of cardinality four only.

Table 2. Translation of the notation of dumont and bäck [31] to our simpler notation

	Our abstract notation	Notation of Dumont and Bäck [31]
Conditions (Inputs)	X_1	<i>EV ERLOST</i>
	X_2	<i>OPPOSITE ELECTORAL FORTUNES</i>
	X_3	<i>MAIN LEFT PARTY IN GOV ERNMENT</i>
	X_4	<i>SURPLUS MAJORITY GOV ERNMENT</i>
	X_5	<i>DISTANCE TO THE MAIN LEFT PARTY</i>
	X_6	<i>DISTANCE TO THE FORMATEUR PARTY</i>
	X_7	<i>PARLIMENTARY EXPERIENCE</i>
	X_8	<i>PROPORTION OF SEATS</i>
Outcome	Z	<i>GREEN PARTY IN GOV ERNMENT</i>

Table 3. Hexadecimal ‘digits’ in terms of decimal digits and binary bits

	Hexadecimal ‘Digits’	Decimal Digits	Binary Bits
0		0	0000
1		1	0001
2		2	0010
3		3	0011
4		4	0100
5		5	0101
6		6	0110
7		7	0111
8		8	1000
9		9	1001
A		10	1010
B		11	1011
C		12	1100
D		13	1101
E		14	1110
F		15	1111

In the rest of this section, we do not try to cover the map immediately as we did in Figs. 4 and 5, but we first decide which set of supporting variables to use by exploring whether specific variables should be supporting variables, and how they might be selected as such. Fig. 6 indicates entry contradiction of the two cells FB and BB (highlighted in yellow), which are mirror images w.r.t. the X_2 border. This asserts that Z cannot be made independent of X_2 , and hence X_2 must be a member of any set of supporting variables of Z . No variable other than X_2 is an essential member of all sets of supporting variables of Z . Fig. 7 imposes conditions for Z in Fig. 3 to make it independent of (the arbitrarily-chosen) X_6 . Cells that are equidistant from the borders of X_6 (highlighted in dark red) are ensured to have equal entries. As a result X_1 , X_4 , and X_8 are added to X_2 as supporting variables for Z . At this stage, we need to check if the resulting set $\{X_1, X_2, X_4, X_8\}$ is a ‘complete’ minimal supporting set for Z . Fig. 8 answers this question in the affirmative, as it now imposes conditions (in red) for Z in Fig. 7 to make it

independent of X_7 , followed by conditions (in blue) to make Z independent of X_3 . It is obvious that Z cannot now be made independent of X_5 . Hence, the set $\{X_1, X_2, X_4, X_5, X_8\}$ is yet another minimal supporting set for f with the following single IDF

$$Z = X_1 X_2 (X_8 \vee X_4 X_5) = X_1 X_2 X_8 \vee X_1 X_2 X_4 X_5. \quad (7)$$

While Figs. 7 and 8 enabled us to obtain this third solution in (7), the forthcoming Figs. 9 and 10 yield a fourth solution by pursuing an alternative scheme. Fig. 9 imposes conditions for Z in Fig. 3 to make it independent of (again, arbitrarily-chosen) X_1 . Cells that are equidistant from the borders of X_1 (highlighted in red) are ensured to have equal entries. As a result X_3, X_4, X_6 , and X_8 are added to X_2 as supporting variables for Z . At this stage, we have to consider the proposition that the set $\{X_2, X_3, X_4, X_6, X_8\}$ is another ‘complete’ minimal supporting set for Z . This proposition turns out to be true with the aid of Fig.10. This figure imposes conditions (in red) for Z in Fig. 9 to make it independent of X_5 ,

		X_3							X_1									
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8	
X_7	0			0	0	0	0							0				
	1					0												
	3					0				0					0			
	2											0						
	6												0					
	7																	
	5																	
	4			0		0												
	C																	
	D													0				
X_5	X_7	F										1	1					
		E										1		0				
		A	0		0			0					0					
		B					0		0	0			1		0	0		
	9					0		0										
	8	0		0		0		0										
			X_4				X_4				X_4				X_4			
			X_2															

Z

Fig. 6. Entry contradiction of the two cells FB and BB (highlighted in yellow), which are mirror images w.r.t. the X_2 border. This asserts that Z cannot be made independent of X_2 , and hence X_2 must be a member of any set of supporting variables of Z. No variable other than X_2 is an essential member of all sets of supporting variables of Z

		X_3							X_1									
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8	
X_7	0			0	0	0	0							0				
	1					0												
	3					0				0					0			
	2											0						
	6												0					
	7					0				0					0	0		
	5					0												
	4			0	0	0	0							0				
	C	0		0		0				0								
	D					0		0							0			
X_5	X_7	F				0	0	0				1	1	0	0		0	
		E	0		0		0					1	0	0				
		A	0		0			0				1	0	0				
		B					0		0	0			1	1	0	0		
	9					0		0						0				
	8	0		0		0		0										
			X_4				X_4				X_4				X_4			
			X_2															

Z

Fig. 7. Imposing conditions for Z in Fig. 3 to make it independent of X_6 . Cells that are equidistant from the borders of X_6 (highlighted in dark red) are ensured to have equal entries. As a result, X_1 , X_4 , and X_8 are added to X_2 as supporting variables for Z. We need to check if the set $\{X_1, X_2, X_4, X_8\}$ is yet another 'complete' minimal supporting set for Z

												X ₁					
		X ₃										X ₃					
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8
X ₅	X ₇	0			0	0	0	0				0	0	0	0		
		1					0					0					
		3	0		0		0			0	0		0		0		0
		2					0						0				
		6															
	X ₇	7															
		5															
		4			0		0						0		0		
		C															
		D					0								0		
X ₇	F					1	1				1	1					
	E					0		1			1		0				
	A	0		0		0	0				0	0		0		0	
	B	0		0	0	0	1	0	0	0	0	1	0	0	0	0	
	9	0					0	0	0	0							
8	0		0		0			0	0			0		0		0	
		X ₄					X ₄					X ₄					
		X ₂															

Z

Fig. 9. Imposing conditions for Z in Fig. 3 to make it independent of X₁. Cells that are equidistant from the borders of X₁ (highlighted in red) are ensured to have equal entries. As a result X₃, X₄, X₆, and X₈ added to X₂ as supporting variables for Z. We need to check if the set {X₂, X₃, X₄, X₆, X₈} in another ‘complete’ minimal supporting set for Z

												X ₁					
		X ₃										X ₃					
		0	1	3	2	6	7	5	4	C	D	F	E	A	B	9	8
X ₅	X ₇	0	0		0	0	0	0		0	0		0	0	0	0	0
		1	0		0	0	0	1	0	0	0	0	1	0	0	0	0
		3	0		0	0	0	0	1	0	0	0	0	1	0	0	0
		2	0		0	0	0	0	0	0	0	0	0	0	0	0	0
		6			0	0	0	0	1				1	0	0	0	
	X ₇	7				0	1	1					1	1	0		
		5				0	1	1					1	1	0		
		4			0	0	0	0	1				1	0	0	0	
		C			0	0	0	0	1				1	0	0	0	
		D				0	1	1					1	1	0		
X ₇	F				0	1	1					1	1	0			
	E			0	0	0	0	1				1	0	0	0		
	A	0		0	0	0	0	0		0	0	0	0	0	0	0	
	B	0		0	0	0	0	1	0	0	0	0	1	0	0	0	
	9	0		0	0	0	0	1	0	0	0	0	1	0	0	0	
8	0		0	0	0	0	0		0	0		0	0	0	0	0	
		X ₄					X ₄					X ₄					
		X ₂															

$$Z = X_2 (X_4 X_6 \vee X_3 X_6 X_8 \vee X_3 X_4 X_8)$$

$$= X_2 X_4 X_6 \vee X_2 X_3 X_6 X_8 \vee X_2 X_3 X_4 X_8$$

Fig. 10. Imposing conditions (in red) for Z in Fig. 9 to make it independent of X₅, followed by conditions (in blue) to make Z independent of X₇. Hence, the set {X₂, X₃, X₄, X₆, X₈} is yet another minimal supporting set for Z with the single IDF shown. This set possesses a cardinality of five, similarly to the set of supporting variables of the solution of Dumont and Bäck [31] in Fig. 4. Though the IDF so obtained is not compact, it cannot be ruled out

5. CONCLUSIONS

We used a regular and modular eight-variable Karnaugh map to explore a large prominent problem of Qualitative Comparative Analysis (QCA). This problem involves an eight-variable partially-defined Boolean function (PDBF), that is dominantly unspecified. Without using the integer-programming approach, we employed a simple map procedure to discover minimal sets of supporting variables from the outset before proceeding to seek IDF representations. We found that the function considered has at least four minimal sets of supporting variables, each of which corresponding to a unique IDF, as deemed most appropriate. One of the obtained IDF is the sole solution offered earlier by automated means. Political scientists are invited to investigate the various solutions obtained herein and explore whether they make sense in the original context.

This paper is part of an ongoing activity striving to streamline the use of Boolean minimization techniques in QCA applications. Our thesis (in agreement with that in [8] is that these techniques cannot be borrowed without modification from the area of digital design for use in QCA context. A more suitable paradigm for QCA problems (that has long been known [30] is to identify all minimal sets of supporting variables, and then obtain all irredundant disjunctive forms (IDFs) for each of these sets. Such a paradigm usually defeats the purpose of QCA, which seeks a unique answer (irrespective of whether it is justified or not). In fact, the question of the nature of a QCA Boolean function is still highly intriguing and debatable. It seems to us that QCA would be trusted only under a very stringent condition, namely, that it deals with a Boolean function that has a unique minimal set of supporting variables, and whose prime implicants are all essential, i.e., a function which possesses only a single irredundant disjunctive form (IDF) that is both a unique minimal sum and the complete sum (the Blake canonical form).

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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