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Research Article

Global Well-Posedness and Analyticity for the Three-Dimensional Incompressible Nematic Liquid Crystal Flows in Scaling Invariant Spaces

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The Cauchy problem for the three-dimensional incompressible flows of liquid crystals in scaling invariant spaces is considered. In this work, we exhibit three results. First, we prove the global well-posedness of mild solution for the system without the supercritical nonlinearity $|\nabla d|^2 d$ when the norms of the initial data are bounded exactly by the minimal value of the viscosity coefficients. Our second result is a proof of the global existence of mild solution in the time dependent spaces for the system including the term $|\nabla d|^2 d$ for small initial data. Lastly, we also get analyticity of the solution.

1. Introduction and Main Results

Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid and those of a solid crystal. A liquid crystal may flow like a liquid, but its molecules may be oriented in a crystal-like way. There are many different types of liquid crystal phases, which can be distinguished based on their different optical properties. One of the most common liquid crystal phases is the nematic, where the molecules have no positional order, but have long-range orientational order. Nematic liquid crystals are aggregates of molecules which possess same orientational order and are made of elongated, rod-like molecules (see [1–3]). The continuum theory of liquid crystals was developed by Ericksen [4] and Leslie [5] during the period of 1958 through 1968, and for more details, see also the book by de Gennes [6]. Since then, there have been remarkable research developments in liquid crystals from both theoretical and applied aspects. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Ossen-Frank theory (for static nematic liquid crystals, see Hardt-Lin-Kinderlehrer [7]) on the analysis of energy minimal configurations of nematic liquid crystals. In this paper, we mainly study two simplified versions of the hydrodynamics of nematic liquid crystals, but still retain most of the interesting mathematical properties of the original Ericksen-Leslie model (see [4, 5]).

In 1989, Lin [8] first proposed the following a simplified three-dimensional Ericksen-Leslie equation modeling incompressible liquid crystal flows

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi = -\theta \operatorname{div} (\nabla d \otimes \nabla d), \\ \partial_t d + u \cdot \nabla d = \nu \left(\Delta d + |\nabla d|^2 d \right), \\ \operatorname{div} u = 0, \\ (u, d)|_{t=0} = (u_0, d_0), \end{cases} \tag{1}$$

where $u \in \mathbb{R}^3$ is the velocity and $d \in \mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$) is the unit-vector field that represents the macroscopic molecular orientations. The scalar function $\Pi \in \mathbb{R}$ is the pressure. The positive constants μ, θ, ν stand for viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Deborah number for the molecular orientation field, respectively. The symbol $\otimes$ denotes the Kronecker tensor product such that

$u \otimes u = (u_i u_j)_{1\le i,j\le 3}$ and $\nabla d e \nabla d$ denotes a matrix whose ij-th entry is $\partial_{x_i} d \cdot \partial_{x_j} d (1 \le i, j \le 3)$. Indeed, $\nabla d e \nabla d = (\nabla d)^T \nabla d$, where $(\nabla d)^T$ denotes the transpose of the 3×3 matrix ∇d. We set $\theta = 1$ since their exact values do not play any role in our analysis.

For system (1), the appearance of the nonlinear term $|\nabla d|^2 d$ with the restriction $|d| = 1$ causes significant mathematical difficulties. In 1990s, Lin and Liu [9, 10] introduced another simplified three-dimensional Ericksen-Leslie equation modeling incompressible liquid crystal flows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi = -\operatorname{div}(\nabla d e \nabla d), \\ \partial_t d + u \cdot \nabla d = \nu(\Delta d - f(d)), \\ \operatorname{div} u = 0, \\ (u, d)|_{t=0} = (u_0, d_0), \end{cases} \tag{2}$$

where $f(d)$ is a vector valued, smooth, and bounded function defined for all $d \in \mathbb{R}^3$.

When d is a constant vector field, systems (1)-(2) reduce to the three-dimensional incompressible Navier-Stokes equations, which is an extremely important system to describe incompressible fluids. It has attracted great interests among many researchers, and there have been many important developments. Leray [11] and Hopf [12] showed the global existence of weak solutions. Fujita and Kato [13] established the local well-posedness for large initial data and the global well-posedness of strong solutions for small initial data in Sobolev space. Similar results have been established in $L^n(\mathbb{R}^n)$ by Kato [14], in critical Besov space $B^{-1+(n/p)}_{p,\infty}(\mathbb{R}^n)(1 < p < \infty)$ by Cannone [15], and in the larger space BMO^{-1} by Koch and Tataru [16]. In 2011, Lei and Lin [17] proved global well-posedness results in a new space $\mathcal{X}^{-1}$ (see Definition 4) if $\|u_0\|_{\mathcal{X}^{-1}} < \mu$. Based on it, Benameur [18] proved a large time decay to the Lei-Lin solution, and Bae [19] presented analyticity of the solution, respectively.

In the past several decades, there are many fruitful results on the analysis of System (1). In certain Besov spaces, Li-Wang [20] obtained the local strong solution with large initial data and the global strong solution with small data. Hineman and Wang [21] established the global well-posedness of system (1) in dimensions three with small initial data $(u_0, \nabla d_0)$ in L^3_{uloc}, where $L^3_{uloc}(\mathbb{R}^3)$ is the space of uniformly locally L^3-integrable functions. We would like to mention that Wang [22] has recently obtained the global (or local) well-posedness of system (1.1) for initial data (u_0, d_0) belonging to possibly the largest space $\mathrm{BMO}^{-1} \times \mathrm{BMO}$ with $\nabla \cdot u_0 = 0$, which is a invariant space under parabolic scaling associated with system (1), with small norms. Tan and Yin [23] established local well-posedness with large initial data and the existence of global strong solution to system (1) with small initial-boundary condition. For system (2), there are also lots of important conclusions when the function $f(d) = (1/\varepsilon^2)(1 - |d|^2)d$ or $f(d)$ is identically zero (see [8–10, 24–26]). Especially, when $f(d) = 0$, although the case is physically irrelevant, one believes it is of interest from the analysis point of view (see [9]). Lin and Liu [9] established the global existence of classical solutions under the additional assumption that the initial data are small in a suitable sense. Hu-Wang [25] obtained the existence and uniqueness of the global strong solution with small initial data. It should be emphasized that the norms in corresponding spaces of the initial data in all these works mentioned above for system (2) are smaller than the viscosity coefficients μ and ν multiplied by a tiny positive constant ε. Then, an interesting question arises, namely, whether it is possible to establish global existence of solutions for the Cauchy problem to the hydrodynamics of nematic liquid crystals in $\mathcal{X}^{-1} \times \mathcal{X}^0$, provided that the norms of the initial data in $\mathcal{X}^{-1} \times \mathcal{X}^0$ are bounded exactly by the minimal value of the viscosity coefficients. The goal of this paper is to give a positive answer to this question for system (1.2) when $f(d) = 0$. On the other hand, for the more complicated system (1) including the super critical nonlinearity $|\nabla d|^2 d$, we also prove the the global existence of mild solution in the time dependent spaces for small initial data in $\mathcal{X}^{-1} \times \mathcal{X}^0$. Furthermore, we also prove analyticity of mild solution to system (1). Compared with the known results for the incompressible Navier-Stokes equations [17–19], the main difficulty of system (1) is much more complicate nonlinear system due to the super critical nonlinearity $|\nabla d|^2 d$ in the transported heat flow of harmonic map equation and the strong coupling nonlinear term $\mathrm{d}iv(\nabla d e \nabla d)$ in the momentum equation. In particular, in order to avoid trouble by directly taking Fourier transformations for these nonlinear terms, we exploit some important nonlinear estimates in some time dependent spaces.

Our first main result on system (2) then reads as follows:

Theorem 1. *Let $f(d) = 0$ in system (2). Suppose that $(u_0, d_0) \in \mathcal{X}^{-1} \times \mathcal{X}^0$ satisfy*

$$\|u_0\|_{\mathcal{X}^{-1}} + \|d_0\|_{\mathcal{X}^0} < \min\{\mu, \nu\}. \tag{3}$$

Then, system (2) has a unique global-in-time solution

$$(u, d) \in C\left((0, +\infty); \mathcal{X}^{-1} \times \mathcal{X}^0\right) \cap L^1\left((0, +\infty); \mathcal{X}^1 \times \mathcal{X}^2\right). \tag{4}$$

Moreover,

$$\begin{aligned} &\sup_{0 \le t < \infty} \left(\|u(t)\|_{\mathcal{X}^{-1}} + \|d(t)\|_{\mathcal{X}^0}\right) + \left(\mu - \|u_0\|_{\mathcal{X}^{-1}} - \|d_0\|_{\mathcal{X}^0}\right) \int_0^\infty \\ &\quad \cdot \|u(\tau)\|_{\mathcal{X}^1} d\tau + \left(\nu - \|u_0\|_{\mathcal{X}^{-1}} - \|d_0\|_{\mathcal{X}^0}\right) \int_0^\infty \|d(\tau)\|_{\mathcal{X}^2} d\tau \\ &\le \|u_0\|_{\mathcal{X}^{-1}} + \|d_0\|_{\mathcal{X}^0}. \end{aligned} \tag{5}$$

Our second main result in the time dependent spaces on system (1) is the following theorem:

Theorem 2. *Suppose that* $(u_0, d_0) \in \mathcal{X}^{-1} \times \mathcal{X}^0$. *There exists a small enough constant* $\eta > 0$ *such that if*

$$\|u_0\|_{\mathcal{X}^{-1}} + \|d_0\|_{\mathcal{X}^0} \le \eta. \tag{6}$$

Then, system (1) admits a unique global-in-time solution

$$(u, d) \in E\, def = \left(\tilde{L}^\infty\left(X^{-1}\right) \times \tilde{L}^\infty\left(X^0\right)\right) \cap \left(\tilde{L}^1\left(X^1\right) \times \tilde{L}^1\left(X^2\right)\right). \tag{7}$$

Our third main result on the analyticity of solution for system (1) is the following theorem.

Theorem 3. *Under the assumptions in Theorem 2, then the solution for system (1) is analytic in the sense that*

$$\begin{aligned} &\|u\|_{\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^{-1}\right)} + \mu\|u\|_{\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^1\right)} + \|d\|_{\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^0\right)} \\ &\quad + \nu\|d\|_{\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^2\right)} \le C\left(\|u_0\|_{\mathcal{X}^{-1}} + \|d_0\|_{\mathcal{X}^0}\right). \end{aligned} \tag{8}$$

The rest of the paper unfolds as follows. In the next section, we recall some basic notions and useful properties of function spaces. In Section 3, we will present the proof of Theorem 1. Section 4 is devoted to the proof of Theorem 2. At last, we show the analyticity of the solution to system (1).

2. Preliminaries

In this section, we introduce some common notations and basic theories about function spaces, and present some auxiliary lemmas.

Definition 4. For $\alpha \in \mathbb{R}$, we define the function spaces $\mathcal{X}^\alpha$ to be

$$\mathcal{X}^\alpha = \left\{f \in \mathcal{D}'\left(\mathbb{R}^3\right) : \|f\|_{\mathcal{X}^\alpha} < \infty\right\} \tag{9}$$

with

$$\|f\|_{\mathcal{X}^\alpha}\, def = \int_{\mathbb{R}^3} |\xi|^\alpha \left|\widehat{f}\right| d\xi, \tag{10}$$

where $\mathcal{D}'(\mathbb{R}^3)$ represents the space of distributions and $\widehat{f}$ represents the Fourier transformation of f.

By a straight computation, we have

$$\ln_l \frac{\|f(lx)\|_{\mathcal{X}^\alpha}}{\|f(x)\|_{\mathcal{X}^\alpha}} = \ln_l l^\alpha = \alpha. \tag{11}$$

For all $l > 0$, systems (1)-(2) are invariant under the following transformations:

$$\tilde{u} = lu\left(l^2 t, lx\right), \tilde{\Pi} = l^2 \Pi\left(l^2 t, lx\right), \tilde{d} = d\left(l^2 t, lx\right). \tag{12}$$

We say that a function space is the initial critical space for systems (1)-(2) if the associated norm is invariant under the transformation $(u_0, d_0) \longrightarrow (\tilde{u}_0, \tilde{d}_0) \coloneqq (lu_0(lx), d_0(lx))$ for all $l > 0$. Obviously, $\mathcal{X}^{-1} \times \mathcal{X}^0$ is the critical space for systems (1.1)-(1.2).

In what follows, we present the following time dependent function spaces.

Definition 5. For $\alpha \in \mathbb{R}$, we define the function spaces $\tilde{L}^\infty(X^\alpha)$ and $\tilde{L}^1(X^\alpha)$ to be, respectively:

$$\tilde{L}^\infty(X^\alpha) = \left\{f \in \mathcal{D}'\left(\mathbb{R}_+ \times \mathbb{R}^3\right) : \|f\|_{\tilde{L}^\infty(X^\alpha)} < \infty\right\}, \tag{13}$$

with

$$\begin{aligned} &\|f\|_{\tilde{L}^\infty(X^\alpha)}\, def = \int_{\mathbb{R}^3} \left[\sup_{0 \le t < \infty} |\xi|^\alpha \left|\widehat{f}(t, \xi)\right|\right] d\xi, \\ &\tilde{L}^1(X^\alpha) = \left\{f \in \mathcal{D}'\left(\mathbb{R}_+ \times \mathbb{R}^3\right) : \|f\|_{\tilde{L}^1(X^\alpha)} < \infty\right\}, \end{aligned} \tag{14}$$

with

$$\|f\|_{\tilde{L}^1(X^\alpha)}\, def = \int_{\mathbb{R}^3} \left[\int_0^\infty |\xi|^\alpha \left|\widehat{f}(t, \xi)\right| dt\right] d\xi. \tag{15}$$

Definition 6. For $\alpha \in \mathbb{R}$, we define the function spaces $\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^\alpha)$ and $\tilde{L}^1(e^{\sqrt{t}\Lambda}X^\alpha)$ to be, respectively:

$$\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^\alpha\right) = \left\{f \in \mathcal{D}'\left(\mathbb{R}_+ \times \mathbb{R}^3\right) : \left\|e^{\sqrt{t}\Lambda}f\right\|_{\tilde{L}^\infty(X^\alpha)} < \infty\right\}, \tag{16}$$

with

$$\begin{aligned} &\left\|e^{\sqrt{t}\Lambda}f\right\|_{\tilde{L}^\infty(X^\alpha)}\, def = \int_{\mathbb{R}^3} \left[\sup_{0 \le t < \infty} e^{\sqrt{t}|\xi|} |\xi|^\alpha \left|\widehat{f}(t, \xi)\right|\right] d\xi, \\ &\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^\alpha\right) = \left\{f \in \mathcal{D}'\left(\mathbb{R}_+ \times \mathbb{R}^3\right) : \left\|e^{\sqrt{t}\Lambda}f\right\|_{\tilde{L}^1(X^\alpha)} < \infty\right\}, \end{aligned} \tag{17}$$

with

$$\left\|e^{\sqrt{t}\Lambda}f\right\|_{\tilde{L}^1(X^\alpha)}\, def = \int_{\mathbb{R}^3} \left[\int_0^\infty e^{\sqrt{t}|\xi|} |\xi|^\alpha \left|\widehat{f}(t, \xi)\right| dt\right] d\xi. \tag{18}$$

Here, $e^{\sqrt{t}\Lambda}$ is a Fourier multiplier whose symbol is given by $e^{\sqrt{t}|\xi|}$.

We next present some important properties of the spaces mentioned the above, which will be frequently used in this paper.

Lemma 7 [27].

(i) Let $f, g \in \mathcal{X}^0$, *then* $fg \in \mathcal{X}^0$ *and*

$$\|fg\|_{\mathcal{X}^0} \le \|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^0}. \tag{19}$$

(ii) Let $f \in \mathcal{X}^\alpha \cap \mathcal{X}^{\alpha+2}$, then $f \in \mathcal{X}^{\alpha+1}$ and

$$\|f\|_{\mathcal{X}^{\alpha+1}} \le \|f\|^{1/2}_{\mathcal{X}^\alpha}\|g\|^{1/2}_{\mathcal{X}^{\alpha+2}}. \tag{20}$$

(iii) Let $f \in \mathcal{X}^{-1} \cap \mathcal{X}^1$, $g \in \mathcal{X}^{-1} \cap \mathcal{X}^1$, then $fg \in \mathcal{X}^0$ and

$$\|fg\|_{\mathcal{X}^0} \le \frac{1}{2}\left(\|f\|_{\mathcal{X}^{-1}}\|g\|_{\mathcal{X}^1} + \|f\|_{\mathcal{X}^1}\|g\|_{\mathcal{X}^{-1}}\right). \tag{21}$$

In particular, if $f = g$, we have

$$\left\|f^2\right\|_{\mathcal{X}^0} \le \|f\|_{\mathcal{X}^{-1}}\|f\|_{\mathcal{X}^1}. \tag{22}$$

(iv) If $D^\beta f \in \mathcal{X}^\alpha$, then $f \in \mathcal{X}^{\alpha+\beta}$ and

$$\left\|D^\beta f\right\|_{\mathcal{X}^\alpha} = \|f\|_{\mathcal{X}^{\alpha+\beta}}. \tag{23}$$

For the time dependent function spaces, we have the following properties.

Lemma 8 [27].

(i) Let $f \in \tilde{L}^\infty(X^0)$, $g \in \tilde{L}^1(X^0)$, then $fg \in \tilde{L}^1(X^0)$ and

$$\|fg\|_{\tilde{L}^1(X^0)} \le \|f\|_{\tilde{L}^\infty(X^0)}\|g\|_{\tilde{L}^1(X^0)}. \tag{24}$$

(ii) Let $f \in \tilde{L}^\infty(X^{-1}) \cap \tilde{L}^1(X^1)$, $g \in \tilde{L}^\infty(X^{-1}) \cap \tilde{L}^1(X^1)$, then $fg \in \tilde{L}^1(X^0)$ and

$$\|fg\|_{\tilde{L}^1(X^0)} \le \frac{1}{2}\left(\|f\|_{\tilde{L}^\infty(X^{-1})}\|g\|_{\tilde{L}^1(X^1)} + \|f\|_{\tilde{L}^1(X^1)}\|g\|_{\tilde{L}^\infty(X^{-1})}\right). \tag{25}$$

In particular, if $f = g$ we have

$$\left\|f^2\right\|_{\tilde{L}^1(X^0)} \le \|f\|_{\tilde{L}^\infty(X^{-1})}\|f\|_{\tilde{L}^1(X^1)}. \tag{26}$$

(iii) If $D^\beta f \in \tilde{L}^\infty(X^\alpha)$, then $f \in \tilde{L}^\infty(X^{\alpha+\beta})$ and

$$\left\|D^\beta f\right\|_{\tilde{L}^\infty(X^\alpha)} = \|f\|_{\tilde{L}^\infty(X^{\alpha+\beta})}. \tag{27}$$

(iv) If $D^\beta f \in \tilde{L}^1(X^\alpha)$, then $f \in \tilde{L}^1(X^{\alpha+\beta})$ and

$$\left\|D^\beta f\right\|_{\tilde{L}^1(X^\alpha)} = \|f\|_{\tilde{L}^1(X^{\alpha+\beta})}. \tag{28}$$

Lemma 9 [27].

(i) Let $f \in \tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^0)$, $g \in \tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)$, then $fg \in \tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)$, and there exists a positive constant C such that

$$\|fg\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)} \le C\|f\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^0)}\|g\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)}. \tag{29}$$

(ii) Let $f \in \tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{-1}) \cap \tilde{L}^1(e^{\sqrt{t}\Lambda}X^1)$, $g \in \tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{-1}) \cap \tilde{L}^1(e^{\sqrt{t}\Lambda}X^1)$, then $fg \in \tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)$, and there exists a positive constant C such that

$$\begin{aligned}&\|fg\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)}\\ &\le C\left(\|f\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{-1})}\|g\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^1)} + \|f\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^1)}\|g\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{-1})}\right).\end{aligned} \tag{30}$$

In particular, if $f = g$ we have

$$\left\|f^2\right\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^0)} \le C\|f\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{-1})}\|f\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^1)}. \tag{31}$$

(iii) If $D^\beta f \in \tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^\alpha)$, then $f \in \tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{\alpha+\beta})$ and

$$\left\|D^\beta f\right\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^\alpha)} = \|f\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{\alpha+\beta})}. \tag{32}$$

(iv) If $D^\beta f \in \tilde{L}^1(e^{\sqrt{t}\Lambda}X^\alpha)$, then $f \in \tilde{L}^1(e^{\sqrt{t}\Lambda}X^{\alpha+\beta})$ and

$$\left\|D^\beta f\right\|_{\tilde{L}^1(e^{\sqrt{t}\Lambda}X^\alpha)} = \|f\|_{\tilde{L}^\infty(e^{\sqrt{t}\Lambda}X^{\alpha+\beta})}. \tag{33}$$

Lemma 10 [27]. *Let u satisfy*

$$\begin{cases}\partial_t u - \kappa\Delta u = f,\\ u|_{t=0} = u_0,\end{cases} \tag{34}$$

with $\kappa > 0$, $f \in \tilde{L}^1(X^\alpha)$ and $u_0 \in \mathcal{X}^\alpha$. Then, for all $T > 0$ and a positive constant C, the following a priori estimate is fulfilled:

$$\|u\|_{\tilde{L}^\infty(X^\alpha)} + \kappa\|u\|_{\tilde{L}^1(X^{\alpha+2})} \le C\left(\|u_0\|_{\mathcal{X}^\alpha} + \|f\|_{\tilde{L}^1(X^\alpha)}\right). \quad (35)$$

Lemma 11 [27]. *Let u be a solution to system (34) with $\kappa > 0$, $f \in \tilde{L}^1(e^{\sqrt{t}\Lambda}X^\alpha)$, and $v_0 \in \mathcal{X}^\alpha$. Then, for all $T > 0$ and a positive constant C, the following a priori estimate is fulfilled:*

$$\|u\|_{\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^\alpha\right)} + \kappa\|u\|_{\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^{\alpha+2}\right)} \le C\left(\|u_0\|_{\mathcal{X}^\alpha} + \|f\|_{\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^\alpha\right)}\right). \quad (36)$$

3. The Proof of Theorem 1

In this section, we prove the proof of Theorem 1 and divide it into several steps.

Step 1. The approximate solution sequence. Let ρ be the standard mollifier in $\mathbb{R}^3$: $\rho \in C_0^\infty$, $0 \le \rho \le 1$, $\int \rho(x)dx = 1$. For $\lambda > 0$, let $\rho^\lambda(x) = \lambda^{-3}\rho(\lambda^{-1}x)$ and $u_0^\lambda = \rho^\lambda * u_0, d_0^\lambda = \rho^\lambda * d_0$. For $u_0 \in \mathcal{X}^{-1}, d_0 \in \mathcal{X}^0$, since $|\widehat{\rho}(\xi)| \le \int \rho(x)dx = 1$, one has

$$\begin{cases} \left\|u_0^\lambda\right\|_{\mathcal{X}^{-1}} = \int |\xi|^{-1}\left|\widehat{u_0}(\xi)\right|\left|\widehat{\rho}(\lambda\xi)\right|d\xi \le \|u_0\|_{\mathcal{X}^{-1}}, \\ \left\|d_0^\lambda\right\|_{\mathcal{X}^0} = \int\left|\widehat{d_0}(\xi)\right|\left|\widehat{\rho}(\lambda\xi)\right|d\xi \le \|d_0\|_{\mathcal{X}^0}, \\ \left\|u_0^\lambda\right\|_{L^\infty} \lesssim \int|\xi|\left|\widehat{\rho}(\lambda\xi)\right||\xi|^{-1}\left|\widehat{u_0}(\xi)\right|d\xi \le C_\lambda\|u_0\|_{\mathcal{X}^{-1}}, \\ \left\|d_0^\lambda\right\|_{L^\infty} \lesssim \int\left|\widehat{\rho}(\lambda\xi)\widehat{d_0}(\xi)\right|d\xi \le C_\lambda\|d_0\|_{\mathcal{X}^0}. \end{cases} \quad (37)$$

Thus, by slight modifications of the proof of Theorem 3 in [28] or Theorem 2 in [29], we can obtain a unique local existence for smooth solution (u^λ, d^λ) on some time internal $[0, T_\lambda)$ for the liquid crystal flow system. Furthermore,

$$\begin{cases} \left\|u^\lambda\right\|_{\mathcal{X}^{-1}} = \int_{|\xi|\le1}|\xi|^{-1}\left|\widehat{v^\lambda}\right|d\xi + \int_{|\xi|>1}|\xi|^{-2}\left|\widehat{\sqrt{-\Delta}v^\lambda}\right|d\xi \le C\left\|v^\lambda\right\|_{H^1}, \\ \left\|d^\lambda\right\|_{\mathcal{X}^0} = \int_{|\xi|\le1}\left|\widehat{v^\lambda}\right|d\xi + \int_{|\xi|>1}|\xi|^{-2}\left|\widehat{(-\Delta)v^\lambda}\right|d\xi \le C\left\|v^\lambda\right\|_{H^2} \end{cases}$$
$$\begin{cases} \int_0^{T_\lambda}\left\|v^\lambda(t)\right\|_{\mathcal{X}^1}dt = \int_0^{T_\lambda}\int_{|\xi|\le1}|\xi|\left|\widehat{v^\lambda}\right|d\xi dt + \int_0^{T_\lambda}\int_{|\xi|>1}|\xi|^{-2}\left|\widehat{\sqrt{-\Delta}^3v^\lambda}\right|d\xi dt \le C\int_0^{T_\lambda}\left\|v^\lambda\right\|_{H^3}ds, \\ \int_0^{T_\lambda}\left\|d^\lambda(t)\right\|_{\mathcal{X}^2}dt = \int_0^{T_\lambda}\int_{|\xi|\le1}|\xi|^2\left|\widehat{d^\lambda}\right|d\xi dt + \int_0^{T_\lambda}\int_{|\xi|>1}|\xi|^{-2}\left|\widehat{\sqrt{-\Delta}^4v^\lambda}\right|d\xi dt \le C\int_0^{T_\lambda}\left\|v^\lambda\right\|_{H^4}ds. \end{cases} \quad (38)$$

Therefore, $(u^\lambda, d^\lambda) \in L^\infty(0, T_\lambda; \mathcal{X}^{-1} \cap \mathcal{X}^0) \cap L^1(0, T_\lambda; \mathcal{X}^1 \cap \mathcal{X}^2)$.

Step 2. Global uniform estimates

Taking the Fourier transformations of system (2), we get

$$\begin{cases} \partial_t\widehat{u^\lambda} - i\xi\widehat{\Pi^\lambda} + \mu|\xi|^2\widehat{v^\lambda} = -i\xi\left(\widehat{u^\lambda \otimes u^\lambda}\right) - i\xi\left(\widehat{\nabla d^\lambda e \nabla d^\lambda}\right), \\ \partial_t\widehat{d^\lambda} + \nu|\xi|^2\widehat{d^\lambda} = -\left(\widehat{u^\lambda \cdot \nabla d^\lambda}\right), \\ \xi \cdot \widehat{u^\lambda} = 0. \end{cases} \quad (39)$$

It follows from (39) and Lemma 7(i) (iii) and (iv) that

$$\begin{aligned} &\frac{d}{dt}\left(\left\|u^\lambda\right\|_{\mathcal{X}^{-1}} + \left\|d^\lambda\right\|_{\mathcal{X}^0}\right) + \mu\left\|u^\lambda\right\|_{\mathcal{X}^1} + \nu\left\|d^\lambda\right\|_{\mathcal{X}^2} \\ &\le \left\|u^\lambda \otimes u^\lambda\right\|_{\mathcal{X}^0} + \left\|\nabla d^\lambda e \nabla d^\lambda\right\|_{\mathcal{X}^0} + \left\|u^\lambda \cdot \nabla d^\lambda\right\|_{\mathcal{X}^0} \\ &\le \left\|u^\lambda\right\|_{\mathcal{X}^{-1}}\left\|u^\lambda\right\|_{\mathcal{X}^1} + \left\|\nabla d^\lambda\right\|_{\mathcal{X}^{-1}}\left\|\nabla d^\lambda\right\|_{\mathcal{X}^1} \\ &\quad + \left\|u^\lambda\right\|_{\mathcal{X}^{-1}}\left\|\nabla d^\lambda\right\|_{\mathcal{X}^1} + \left\|u^\lambda\right\|_{\mathcal{X}^1}\left\|\nabla d^\lambda\right\|_{\mathcal{X}^{-1}} \\ &\le \left\|u^\lambda\right\|_{\mathcal{X}^{-1}}\left\|u^\lambda\right\|_{\mathcal{X}^1} + \left\|d^\lambda\right\|_{\mathcal{X}^0}\left\|d^\lambda\right\|_{\mathcal{X}^2} + \left\|u^\lambda\right\|_{\mathcal{X}^{-1}}\left\|d^\lambda\right\|_{\mathcal{X}^2} \\ &\quad + \left\|u^\lambda\right\|_{\mathcal{X}^1}\left\|d^\lambda\right\|_{\mathcal{X}^0}. \end{aligned} \quad (40)$$

Note that

$$\left\|u^{\lambda}(t)\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda}(t)\right\|_{\mathcal{X}^{0}}<\min\{\mu,\nu\}, \quad (41)$$

for $t\in[0,\delta]$ with $0<\delta<T_\lambda$. Therefore,

$$\frac{d}{dt}\left(\left\|u^{\lambda}\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda}\right\|_{\mathcal{X}^{0}}\right)\leq 0, \text{ for } t\in[0,\delta]. \quad (42)$$

Thus, we obtain from (37)

$$\begin{aligned}\left\|u^{\lambda}\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda}\right\|_{\mathcal{X}^{0}}&\leq\left\|u_0^{\lambda}\right\|_{\mathcal{X}^{-1}}+\left\|d_0^{\lambda}\right\|_{\mathcal{X}^{0}}\\&\leq\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}<\min\{\mu,\nu\}.\end{aligned} \quad (43)$$

From a continuity argument in the time variable, we have

$$\left\|u^{\lambda}\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda}\right\|_{\mathcal{X}^{0}}\leq\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}<\min\{\mu,\nu\}, \quad (44)$$

for all $t\in[0,T_\lambda]$. From (40), we get for $\forall t\in[0,T_\lambda]$

$$\begin{aligned}&\left\|u^{\lambda}\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda}\right\|_{\mathcal{X}^{0}}+\left(\mu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}\right)\left\|u^{\lambda}\right\|_{L^1([0,t];\mathcal{X}^1)}\\&\quad+\left(\nu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}\right)\left\|d^{\lambda}\right\|_{L^1([0,t];\mathcal{X}^2)}\leq\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}},\end{aligned} \quad (45)$$

which together with (20) and (45) implies that

$$\int_0^{T_\lambda}\left\|\nabla u^{\lambda}(t)\right\|_{L^\infty}dt\leq\int_0^{T_\lambda}\left\|u^{\lambda}(t)\right\|_{\mathcal{X}^1}dt\leq\frac{\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}}{\mu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}}, \quad (46)$$

$$\begin{aligned}\int_0^{T_\lambda}\left\|\nabla d^{\lambda}(t)\right\|_{L^\infty}^2dt&\leq\int_0^{T_\lambda}\left\|d^{\lambda}(t)\right\|_{\mathcal{X}^1}^2dt\\&\leq\int_0^{T_\lambda}\left\|d^{\lambda}(t)\right\|_{\mathcal{X}^0}\left\|d^{\lambda}(t)\right\|_{\mathcal{X}^2}dt\\&\leq\left\|d^{\lambda}\right\|_{L^\infty([0,T_\lambda];\mathcal{X}^0)}\int_0^{T_\lambda}\left\|d^{\lambda}(t)\right\|_{\mathcal{X}^2}dt\\&\leq\frac{\left(\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}\right)^2}{\nu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}},\end{aligned} \quad (47)$$

where we have used $\|\nabla g\|_{L^\infty}=\|F^{-1}(i\xi\widehat{g})\|_{L^\infty}\leq\|i\xi\widehat{g}\|_{L^1}=\|g\|_{\mathcal{X}^1}$ by the definition of the inverse Fourier transformation.

On the other hand, the standard energy method in [30] gives that

$$\begin{aligned}\left\|u^{\lambda}(t)\right\|_{H^s}+\left\|d^{\lambda}(t)\right\|_{H^{s+1}}&\leq\left(\|u_0\|_{H^s}+\|d_0\|_{H^{s+1}}\right)\exp\\&\cdot\left\{c_s\left(\int_0^{T_\lambda}\left\|\nabla u^{\lambda}(s)\right\|_{L^\infty}+\left\|\nabla d^{\lambda}(s)\right\|_{L^\infty}^2\right)ds\right\}.\end{aligned} \quad (48)$$

Putting (46) and (47) into (48), we obtain

$$\begin{aligned}&\left\|u^{\lambda}(t)\right\|_{H^s}+\left\|d^{\lambda}(t)\right\|_{H^{s+1}}\leq\left(\|u_0\|_{H^s}+\|d_0\|_{H^{s+1}}\right)\exp\\&\cdot\left\{c_s\min\left\{\frac{\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}}{\mu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}},\frac{\left(\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}\right)^2}{\nu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}}\right\}\right\},\end{aligned} \quad (49)$$

for all $0\leq t<T_\lambda$ and all $s>0$. This implies that $T_\lambda=\infty$. Moreover, we obtain the following global uniform estimates for

$$\begin{aligned}&\sup_{0\leq t<\infty}\left(\left\|u^{\lambda}(t)\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda}(t)\right\|_{\mathcal{X}^{0}}\right)\\&\quad+\left(\mu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}\right)\left\|u^{\lambda}\right\|_{L^1([0,\infty);\mathcal{X}^1)}\\&\quad+\left(\nu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^{0}}\|\right)\left\|d^{\lambda}\right\|_{L^1([0,\infty);\mathcal{X}^2)}\\&\leq\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^{0}}.\end{aligned} \quad (50)$$

The estimate (50) implies that there exists a subsequence of (u^λ,d^λ) (we will still denote it by (u^λ,d^λ)) such that as $\lambda\longrightarrow 0$,

$$\begin{aligned}&u^\lambda\rightharpoonup u \text{ in } L^1\left(\mathbb{R}_+;\mathcal{X}^1\right),\quad u^\lambda\rightharpoonup u \text{ weakly}^* \text{ in } L^\infty\left(\mathbb{R}_+;\mathcal{X}^{-1}\right),\\&d^\lambda\rightharpoonup d \text{ in } L^1\left(\mathbb{R}_+;\mathcal{X}^2\right),\quad d^\lambda\rightharpoonup d \text{ weakly}^* \text{ in } L^\infty\left(\mathbb{R}_+;\mathcal{X}^{0}\right),\end{aligned} \quad (51)$$

for some,

$$(u,d)\in L^\infty\left(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0\right)\cap L^1\left(\mathbb{R}_+;\mathcal{X}^1\times\mathcal{X}^2\right). \quad (52)$$

Step 3. Strong converges

By straight computations, we obtain

$$\begin{aligned}\left\|u_0^\lambda-u_0\right\|_{X^{-1}}&=\int_{|\xi|\leq M}|\xi|^{-1}\left|\widehat{\zeta}(\lambda\xi)-1\right|\left|\widehat{u_0}(\xi)\right|d\xi\\&\quad+\int_{|\xi|>M}|\xi|^{-1}\left|\widehat{\zeta}(\lambda\xi)-1\right|\left|\widehat{u_0}(\xi)\right|d\xi\\&\leq 2\sup_{|\eta|\leq\lambda M}\left|\widehat{\zeta}(\eta)-1\right|\int_{|\xi|<M}|\xi|^{-1}\left|\widehat{u_0}(\xi)\right|d\xi\\&\quad+\int_{|\xi|>M}|\xi|^{-1}\left|\widehat{\zeta}(\lambda\xi)-1\right|\left|\widehat{u_0}(\xi)\right|d\xi,\end{aligned}$$

$$\begin{aligned}\left\|d_0^\lambda-d_0\right\|_{X^{0}}&=\int_{|\xi|\leq M}\left|\widehat{\zeta}(\lambda\xi)-1\right|\left|\widehat{d_0}(\xi)\right|d\xi+\int_{|\xi|>M}\left|\widehat{\zeta}(\lambda\xi)-1\right|\left|\widehat{d_0}(\xi)\right|d\xi\\&\leq 2\sup_{|\eta|\leq\lambda M}\left|\widehat{\zeta}(\eta)-1\right|\int_{|\xi|<M}\left|\widehat{d_0}(\xi)\right|d\xi\\&\quad+\int_{|\xi|>M}\left|\widehat{\zeta}(\lambda\xi)-1\right|\left|\widehat{d_0}(\xi)\right|d\xi.\end{aligned} \quad (53)$$

By taking $M=\lambda^{-1/2}$, and using $\widehat{\zeta}(0)=\int\zeta(x)dx=1$, we conclude that

$$\left\|u_0^\lambda - u_0\right\|_{\mathcal{X}^{-1}} \longrightarrow 0, \left\|d_0^\lambda - d_0\right\|_{\mathcal{X}^0} \longrightarrow 0 \text{ as } \lambda \longrightarrow 0. \tag{54}$$

In what follows, we prove the strong convergences of u^λ and d^λ. Similar to (40), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^0}\right)+\mu\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}+\nu\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^2}\\
&\quad\le\left\|u^{\lambda_1}\otimes\left(u^{\lambda_1}-u^{\lambda_2}\right)\right\|_{\mathcal{X}^0}+\left\|u^{\lambda_2}\otimes\left(u^{\lambda_1}-u^{\lambda_2}\right)\right\|_{\mathcal{X}^0}\\
&\qquad+\left\|\nabla d^{\lambda_1}\mathrm{e}\left(\nabla u^{\lambda_1}-\nabla u^{\lambda_2}\right)\right\|_{\mathcal{X}^0}+\left\|\nabla d^{\lambda_2}\mathrm{e}\left(\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right)\right\|_{\mathcal{X}^0}\\
&\qquad+\left\|u^{\lambda_1}\otimes\left(\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right)\right\|_{\mathcal{X}^0}+\left\|\left(u^{\lambda_1}-u^{\lambda_2}\right)\cdot\nabla d^{\lambda_2}\right\|_{\mathcal{X}^0}\\
&\quad\le\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^1}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^{-1}}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|u^{\lambda_2}\right\|_{\mathcal{X}^1}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|\nabla d^{\lambda_1}\right\|_{\mathcal{X}^1}\left\|\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|\nabla d^{\lambda_1}\right\|_{\mathcal{X}^{-1}}\left\|\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|\nabla d^{\lambda_2}\right\|_{\mathcal{X}^1}\left\|\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|\nabla d^{\lambda_2}\right\|_{\mathcal{X}^{-1}}\left\|\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|u^{\lambda_1}\right\|_{x^1}\left\|\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^{-1}}\left\|\nabla d^{\lambda_1}-\nabla d^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|\nabla d^{\lambda_2}\right\|_{\mathcal{X}^1}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|\nabla d^{\lambda_2}\right\|_{\mathcal{X}^{-1}}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\quad\le\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^1}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^{-1}}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|u^{\lambda_2}\right\|_{\mathcal{X}^1}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}\\
&\qquad+\frac12\left\|d^{\lambda_1}\right\|_{\mathcal{X}^2}\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^0}+\frac12\left\|d^{\lambda_1}\right\|_{\mathcal{X}^0}\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^2}\\
&\qquad+\frac12\left\|d^{\lambda_2}\right\|_{\mathcal{X}^2}\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^0}+\frac12\left\|d^{\lambda_2}\right\|_{\mathcal{X}^0}\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^2}\\
&\qquad+\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^1}\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^0}+\frac12\left\|u^{\lambda_1}\right\|_{\mathcal{X}^{-1}}\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^2}\\
&\qquad+\frac12\left\|d^{\lambda_2}\right\|_{\mathcal{X}^2}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\frac12\left\|d^{\lambda_2}\right\|_{\mathcal{X}^0}\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1},
\end{aligned}\tag{55}$$

which together with (50) yields

$(d/dt)(\|u^{\lambda_1}-u^{\lambda_2}\|_{\mathcal{X}^{-1}}+\|d^{\lambda_1}-d^{\lambda_2}\|_{\mathcal{X}^0})+(\mu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^0})\|u^{\lambda_1}-u^{\lambda_2}\|_{\mathcal{X}^1}+(\nu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^0})\|d^{\lambda_1}-d^{\lambda_2}\|_{\mathcal{X}^2}\le 1/2(\|u^{\lambda_1}\|_{\mathcal{X}^1}+\|u^{\lambda_2}\|_{\mathcal{X}^1}+\|d^{\lambda_2}\|_{\mathcal{X}^2})\|u^{\lambda_1}-u^{\lambda_2}\|_{\mathcal{X}^{-1}}+1/2(\|d^{\lambda_1}\|_{\mathcal{X}^2}+\|d^{\lambda_2}\|_{\mathcal{X}^2}+\|u^{\lambda_1}\|_{\mathcal{X}^1})\|d^{\lambda_1}-d^{\lambda_2}\|_{\mathcal{X}^0}$.(23)

Thus, we obtain

$$\begin{aligned}
&\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^{-1}}+\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^0}\\
&\quad\le\left(\left\|u_0^{\lambda_1}-u_0^{\lambda_1}\right\|_{\mathcal{X}^{-1}}+\left\|d_0^{\lambda_1}-d_0^{\lambda_1}\right\|_{\mathcal{X}^0}\right)\exp\\
&\qquad\cdot\left\{2\frac{\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^0}}{\min\{\mu,\nu\}-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^0}}\right\},\\
&\qquad\cdot(\mu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^0})\int_0^\infty\left\|u^{\lambda_1}-u^{\lambda_2}\right\|_{\mathcal{X}^1}dt\\
&\qquad+\left(\nu-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^0}\right)\int_0^\infty\left\|d^{\lambda_1}-d^{\lambda_2}\right\|_{\mathcal{X}^2}dt\\
&\quad\le\left(\left\|u_0^{\lambda_1}-u_0^{\lambda_1}\right\|_{\mathcal{X}^{-1}}+\left\|d_0^{\lambda_1}-d_0^{\lambda_1}\right\|_{\mathcal{X}^0}\right)\exp\\
&\qquad\cdot\left\{2\frac{\|u_0\|_{\mathcal{X}^{-1}}+\|d_0\|_{\mathcal{X}^0}}{\min\{\mu,\nu\}-\|u_0\|_{\mathcal{X}^{-1}}-\|d_0\|_{\mathcal{X}^0}}\right\}.
\end{aligned}\tag{56}$$

Combining (54) with (56), we conclude that (u^λ, d^λ) is a Cauchy sequence in $L^\infty(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0)\cap L^1(\mathbb{R}_+;\mathcal{X}^1\times\mathcal{X}^2)$ and the convergence in (51) is a strong one. In fact, (56) also yields the uniqueness of solutions in the space $L^\infty(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0)\cap L^1(\mathbb{R}_+;\mathcal{X}^1\times\mathcal{X}^2)$ under the assumption (1.1).

Step 4. Time continuity

To get the further time regularity of $u(t,x)$ and $d(t,x)$, we come back to the Equation (39). We claim that $(\partial_t u^\lambda, \partial_t d^\lambda)$ are uniformly bounded in $L^1(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0)$. In fact, form

$$\begin{cases}\partial_t\widehat{u^\lambda}+\widehat{\nabla\Pi^\lambda}-\mu\widehat{\Delta u^\lambda}=-\nabla\cdot\left(\widehat{u^\lambda\otimes u^\lambda}\right)-\nabla\cdot\left(\widehat{\nabla d^\lambda\odot\nabla d^\lambda}\right),\\ \partial_t\widehat{d^\lambda}-\nu\widehat{\Delta d^\lambda}=-\left(\widehat{u^\lambda\cdot\nabla d^\lambda}\right),\\ \xi\cdot\widehat{u^\lambda}=0,\end{cases}\tag{57}$$

we have

$$\begin{aligned}
&\int_0^\infty\left\|\partial_t u^\lambda\right\|_{\mathcal{X}^{-1}}dt\le\mu\int_0^\infty\left\|\Delta u^\lambda\right\|_{\mathcal{X}^{-1}}dt+\int_0^\infty\left\|\nabla\Pi^\lambda\right\|_{\mathcal{X}^{-1}}dt\\
&\qquad+\int_0^\infty\left\|\nabla\cdot\left(u^\lambda\otimes u^\lambda\right)\right\|_{\mathcal{X}^{-1}}dt+\int_0^\infty\left\|\nabla\cdot\left(\nabla d^\lambda\odot\nabla d^\lambda\right)\right\|_{\mathcal{X}^{-1}}dt\\
&\quad\le C\left(\int_0^\infty\left\|u^\lambda\right\|_{\mathcal{X}^1}dt+\int_0^\infty\left\|u^\lambda\otimes u^\lambda\right\|_{\mathcal{X}^0}dt+\int_0^\infty\left\|\nabla d^\lambda\odot\nabla d^\lambda\right\|_{\mathcal{X}^0}dt\right)\\
&\quad\le C\left(\int_0^\infty\left\|u^\lambda\right\|_{\mathcal{X}^1}dt+\sup_t\left\|u^\lambda\right\|_{\mathcal{X}^{-1}}\int_0^\infty\left\|u^\lambda\right\|_{\mathcal{X}^1}dt+\sup_t\left\|d^\lambda\right\|_{\mathcal{X}^0}\int_0^\infty\right.\\
&\qquad\left.\cdot\left\|d^\lambda\right\|_{\mathcal{X}^2}dt\right),
\end{aligned}\tag{58}$$

and

$$\begin{aligned}
\int_0^\infty\left\|\partial_t d^\lambda\right\|_{\mathcal{X}^0}dt&\le\nu\int_0^\infty\left\|d^\lambda\right\|_{\mathcal{X}^2}dt+\int_0^\infty\left\|u^\lambda\cdot\nabla d^\lambda\right\|_{\mathcal{X}^0}dt\\
&\le C\left(\int_0^\infty\left\|d^\lambda\right\|_{\mathcal{X}^2}dt+\sup_t\left\|u^\lambda\right\|_{\mathcal{X}^{-1}}\int_0^\infty\left\|d^\lambda\right\|_{\mathcal{X}^2}dt\right.\\
&\qquad\left.+\sup_t\left\|d^\lambda\right\|_{\mathcal{X}^0}\int_0^\infty\left\|u^\lambda\right\|_{\mathcal{X}^1}dt\right),
\end{aligned}\tag{59}$$

which implies that

$$\left(\partial_t u^\lambda,\partial_t d^\lambda\right)\in L^1\left(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0\right).\tag{60}$$

(60) allows us to improve (52) and to finally conclude

$$\begin{aligned}&(u,d)\in C\left(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0\right)\cap L^1\left(\mathbb{R}_+;\mathcal{X}^1\times\mathcal{X}^2\right),\\&(\partial_t u,\partial_t d)\in L^1\left(\mathbb{R}_+;\mathcal{X}^{-1}\times\mathcal{X}^0\right).\end{aligned}\tag{61}$$

4. The Proof of Theorem 2

In this section, we show the proof of Theorem 2. Here, we only show the global uniform estimates for (u, d) in E, where the norm of E is defined by

$$\|(u, d)\|_E = \|u\|_{E_1} + \|d\|_{E_2} \tag{62}$$

with

$$\begin{aligned} \|u\|_{E_1} &= \|u\|_{\tilde{L}^\infty(X^{-1})} + \mu\|u\|_{\tilde{L}^1(X^1)}, \\ \|d\|_{E_2} &= \|d\|_{\tilde{L}^\infty(X^0)} + \nu\|d\|_{\tilde{L}^1(X^2)}. \end{aligned} \tag{63}$$

According to $\|u_0\| + \|d_0\| \le \eta$, for some sufficiently small η, there exists a positive constant M which will be chosen later such that

$$\|u, d\|_E \le M\eta. \tag{64}$$

Employing Lemmas 8-10 and the boundness of the Fourier multiplier $\mathbb{P}$, we conclude that

$$\begin{aligned} &\|u\|_{\tilde{L}^\infty(X^{-1})} + \mu\|u\|_{\tilde{L}^1(X^1)} \\ &\quad \le C\left(\|u_0\|_{\mathscr{X}^{-1}} + \|\mathbb{P}\nabla\cdot(u\otimes u+\nabla d\odot\nabla d)\|_{\tilde{L}^1(X^{-1})}\right) \\ &\quad \le C\left(\|u_0\|_{\mathscr{X}^{-1}} + \|u\otimes u+\nabla d\odot\nabla d\|_{\tilde{L}^1(X^0)}\right) \\ &\quad \le C\left(\|u_0\|_{\mathscr{X}^{-1}} + \|u\|_{\tilde{L}^\infty(X^{-1})}\|u\|_{\tilde{L}^1(X^1)} + \|\nabla d\|_{\tilde{L}^\infty(X^{-1})}\|\nabla d\|_{\tilde{L}^1(X^1)}\right) \\ &\quad \le C\left(\|u_0\|_{\mathscr{X}^{-1}} + \|u\|_{\tilde{L}^\infty(X^{-1})}\|u\|_{\tilde{L}^1(X^1)} + \|d\|_{\tilde{L}^\infty(X^0)}\|d\|_{\tilde{L}^1(X^2)}\right) \\ &\quad \le C\left(\eta + M^2\eta^2\right), \end{aligned} \tag{65}$$

$$\begin{aligned} &\|d\|_{\tilde{L}^\infty(X^0)} + \nu\|d\|_{\tilde{L}^1(X^2)} \\ &\quad \le C\left(\|d_0\|_{\mathscr{X}^0} + \|u\cdot\nabla d\|_{\tilde{L}^1(X^0)} + \left\||\nabla d|^2 d\right\|_{\tilde{L}^1(X^0)}\right) \\ &\quad \le C\Big(\|d_0\|_{\mathscr{X}^0} + \|u\|_{\tilde{L}^\infty(X^{-1})}\|\nabla d\|_{\tilde{L}^1(X^1)} + \|u\|_{\tilde{L}^1(X^1)}\|\nabla d\|_{\tilde{L}^\infty(X^{-1})} \\ &\qquad + \|d\|_{\tilde{L}^\infty(X^0)}\left\||\nabla d|^2\right\|_{\tilde{L}^1(X^0)}\Big) \\ &\quad \le C\Big(\|d_0\|_{\mathscr{X}^0} + \|u\|_{\tilde{L}^\infty(X^{-1})}\|d\|_{\tilde{L}^1(X^2)} + \|u\|_{\tilde{L}^1(X^1)}\|d\|_{\tilde{L}^\infty(X^0)} \\ &\qquad + \|d\|_{\tilde{L}^\infty(X^0)}\|\nabla d\|_{\tilde{L}^\infty(X^{-1})}\|\nabla d\|_{\tilde{L}^1(X^1)}\Big) \\ &\quad \le C\Big(\|d_0\|_{\mathscr{X}^0} + \|u\|_{\tilde{L}^\infty(X^{-1})}\|d\|_{\tilde{L}^1(X^2)} + \|u\|_{\tilde{L}^1(X^1)}\|d\|_{\tilde{L}^\infty(X^0)} \\ &\qquad + \|d\|_{\tilde{L}^\infty(X^0)}\|d\|_{\tilde{L}^\infty(X^0)}\|d\|_{\tilde{L}^1(X^2)}\Big) \le C\left(\eta + M^2\eta^2 + M^3\eta^3\right). \end{aligned} \tag{66}$$

Combining (65) with (66), we deduce that

$$\|u, d\|_E \le C\left(\eta + M^2\eta^2 + M^3\eta^3\right). \tag{67}$$

Taking $M = 4C$ and then choosing η small enough such that

$$M^2\eta \le \frac{1}{2}, \quad M^3\eta^2 \le \frac{1}{2}, \tag{68}$$

then we infer that

$$\|u, d\|_E \le \frac{1}{2}M\eta, \tag{69}$$

which implies that $(u, d) \in (\tilde{L}^\infty(X^{-1}) \times \tilde{L}^\infty(X^0)) \cap (\tilde{L}^1(X^1) \times \tilde{L}^1(X^2))$.

At last, we consider the following iteration scheme:

$$\begin{aligned} u^{(k+1)}(t,x) &= u^{(1)} - \int_0^t e^{-\mu\Delta(t-s)}\mathbb{P}\left(u^{(k)}\cdot\nabla u^{(k)} + \nabla\cdot\left(\nabla d^{(k)} \mathrm{e} \nabla d^{(k)}\right)\right)ds, \\ d^{(k+1)}(t,x) &= d^{(1)} - \int_0^t e^{-\nu\Delta(t-s)}\left(u^{(k)}\cdot\nabla d^{(k)} - \left|\nabla d^{(k)}\right|^2 d^{(k)}\right)ds, \\ u^{(1)} &= e^{-\mu\Delta t}u_0, \\ d^{(1)} &= e^{-\nu\Delta t}d_0. \end{aligned} \tag{70}$$

Based on (64), we can also get the global uniform estimates for $(u^{(k)}, d^{(k)})$ in E. Due to the similarity to the Steps 3–4, here, we omit the remaining part of the proof.

5. The Proof of Theorem 3

In this section, we show the proof of Theorem 3. Here, we only present a priori estimate (8) and skip the iteration step. Introducing the following function spaces

$$\mathbb{E} \stackrel{\text{def}}{=} \left(\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^{-1}\right) \times \tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^0\right)\right) \cap \left(\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^1\right) \times \tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^2\right)\right). \tag{71}$$

The norm of $\mathbb{E}$ is defined by

$$\|u, d\|_{\mathbb{E}} = \|u\|_{\mathbb{E}_1} + \|d\|_{\mathbb{E}_2}, \tag{72}$$

with

$$\begin{aligned} \|u\|_{\mathbb{E}_1} &= \|u\|_{\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^{-1}\right)} + \mu\|u\|_{\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^1\right)}, \\ \|d\|_{\mathbb{E}_2} &= \|d\|_{\tilde{L}^\infty\left(e^{\sqrt{t}\Lambda}X^0\right)} + \nu\|d\|_{\tilde{L}^1\left(e^{\sqrt{t}\Lambda}X^2\right)}. \end{aligned} \tag{73}$$

Due to the small initial conditions $\|u_0\| + \|d_0\| \le \eta$, there exists a positive constant $\mathbb{M}$ which will be chosen later such that

$$\|u, d\|_{\mathbb{E}} \le \mathbb{M}\eta. \tag{74}$$

Taking the same procedures (65)-(66), and then employing Lemmas 9-11 and the boundness of the Fourier multiplier $\mathbb{P}$, we have

$$\|u, d\|_{\mathbb{E}} \leq \frac{1}{2}\mathbb{M}\eta. \quad (75)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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