Research Article

Initial Value Problems with Generalized Fractional Derivatives and Their Solutions via Generalized Laplace Decomposition Method

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In this article, we use the ρ-Laplace decomposition method to find the solution to the initial value problems that involve generalized fractional derivatives. The ρ-Laplace decomposition method is used to get approximate series solutions. The Adomian decomposition is improved with the assistance of the ρ-Laplace transform to examine the solutions of the given examples to demonstrate the precision of the current technique.

1. Introduction

Fractional calculus has grown in popularity in applied mathematics, and it has been used to describe a variety of physical scientific domains such as viscoelasticity, control, and diffusion. Fractional differential equations are encountered in a variety of areas of engineering and physical study, such as [1–3]. Different approaches have been used to study and solve fractional differential equations [4–15]. The integral transform methods are critical for handling a wide variety of problems. To get analytical solutions, in certain cases, they have been used in conjunction with several analytical techniques [16–19], and it has been shown to be successful in overcoming problems. The ρ-Laplace transform is a generalized Laplace transform introduced by [20] to solve fractional differential equations with generalized fractional derivatives. The influence of the different values of order q and parameter ρ on the solution of a common widespread fractional differential equation with Caputo generalized fractional derivative is studied.

In [21], Shah et al. used the homotopy perturbation method with the ρ-Laplace transform to obtain the solutions of the nonlinear system of fractional Kersten–Krasil'shchik coupled Kdv equation. Thanompolkrang et al. in [22] found the fractional Black–Scholes European equation solution using the homotopy perturbation method with generalized Laplace transform. Bhangale et al. [23] solved fractional differential equations with generalized fractional derivatives by using a coupling of ρ-Laplace transform method and an iterative approach.

The ρ-Laplace transform decomposition method combines the ρ-Laplace transform and a domain decomposition method. The goal of this research is to use the ρ-Laplace transform decomposition technique to solve initial value problems involving generalized fractional derivatives.

This work is broken down as follows. In Section 2, we give the definitions of generalized fractional integral, generalized fractional derivative, and the ρ-Laplace transform. The ρ-Laplace decomposition method and the convergence are shown in Sections 3 and 4, respectively. In Section 5, test examples are examined to illustrate the efficiency and features of the presented method. Finally, conclusions from the present research are given in Section 6.

The proposed method has some advantages and disadvantages:

(i) This method gives the solution with fewer arithmetic calculations and with high efficiency

(ii) The ρ-Laplace decomposition method is devoid of any restrictive assumptions, perturbations, discretization, or linearization

(iii) The main disadvantage of this method is that it gives the solution in a series form, but the series solution needs to be truncated to use it in the real-time applications. Moreover, the rate and region of convergence are issues that could occur

2. Basic Definitions

Definition 1. For the function $\hbar : [0,\infty] \longrightarrow \mathbb{R}$, the generalized fractional integral of order $q > 0$, where $p > 0$, given in [20] is defined by

$$I_a^{q,p}\hbar(\mathcal{X}) = \frac{1}{\Gamma(q)}\int_a^{\mathcal{X}} \left(\frac{\mathcal{X}^p - \mathcal{S}^p}{p}\right)^{q-1} \frac{\hbar(\mathcal{S})}{\mathcal{S}^{1-p}} d\mathcal{S}, \quad \mathcal{X} > a \geq 0, 0 < q < 1. \tag{1}$$

Definition 2. For the function $\hbar : [0,\infty] \longrightarrow \mathbb{R}$, the generalized fractional derivative of order $q > 0$, where $p > 0$, given in [20] is defined by

$$\mathfrak{D}_a^{q,p}\hbar(\mathcal{X}) = \frac{1}{\Gamma(1-q)}\left(\frac{d}{d\mathcal{X}}\right)\int_a^{\mathcal{X}} \left(\frac{\mathcal{X}^p - \mathcal{S}^p}{p}\right)^{-q} \frac{\hbar(\mathcal{S})}{\mathcal{S}^{1-p}} d\mathcal{S}, \quad \mathcal{X} > a \geq 0, 0 < q < 1. \tag{2}$$

Definition 3. For the function $\hbar : [0,\infty] \longrightarrow \mathbb{R}$, the Caputo generalized fractional derivative of order $q > 0$, where $p > 0$, given in [20] is defined by

$$^{GC}\mathfrak{D}_a^{q,p}\hbar(\mathcal{X}) = \frac{1}{\Gamma(1-q)}\int_a^{\mathcal{X}} \left(\frac{\mathcal{X}^p - \mathcal{S}^p}{p}\right)^{-q} \frac{\mathcal{B}^n \hbar(\mathcal{S})}{\mathcal{S}^{1-p}} d\mathcal{S}, \quad \mathcal{X} > a \geq 0, 0 < q < 1, \tag{3}$$

where $\mathcal{B} = \mathcal{X}^{1-\mathcal{B}} d/d\mathcal{X}$.

Definition 4. The p-Laplace transform of a function $\hbar : [0, \infty] \longrightarrow \mathbb{R}$ given in [20] is defined by

$$\mathcal{L}_p\{\hbar(\mathcal{X})\} = \int_0^\infty e^{-\mathcal{S}\mathcal{X}^p/p}\hbar(\mathcal{X})\frac{d\mathcal{X}}{\mathcal{X}^{1-p}}. \tag{4}$$

The p-Laplace transform of the function $\hbar$ with Caputo generalized fractional derivative given in [20] is defined by

$$\mathcal{L}_p\{\mathfrak{D}_a^{q,p}\hbar(\mathcal{X})\} = \mathcal{S}^q \mathcal{L}_p\{\hbar(\mathcal{X})\} - \sum_{k=0}^{n-1}\mathcal{S}^{q-k-1}(I^{q,p}\mathcal{B}^n\hbar)(0). \tag{5}$$

3. Analysis of Method

Consider the fractional problem with Caputo generalized fractional derivative in the form

$$\mathfrak{D}_0^{q,p}\mathcal{Y}(\tau) + \mathcal{M}\mathcal{Y}(\tau) + \mathcal{N}\mathcal{Y}(\tau) = \mathcal{F}(\tau), q > 0, 0 < q \leq 1, \tag{6}$$

subject to

$$\mathcal{Y}(0) = \mathcal{G}(\tau), \tag{7}$$

where $\mathcal{M}\mathcal{Y}(\tau)$ is a linear function, $\mathcal{N}\mathcal{Y}(\tau)$ is a nonlinear function, and $\mathcal{F}(\tau)$ is a function of τ.

Taking p-Laplace transform to Equation (6), we get

$$\mathcal{L}_p\left[\mathfrak{D}_0^{q,p}\mathcal{Y}(\tau) + \mathcal{M}\mathcal{Y}(\tau) + \mathcal{N}\mathcal{Y}(\tau)\right] = \mathcal{L}_p[\mathcal{F}(\tau)], \quad q > 0, 0 < q \leq 1,$$

$$\mathcal{L}_p[\mathcal{Y}(\tau)] = \frac{\mathcal{G}(\tau)}{\mathcal{S}} + \frac{1}{\mathcal{S}^q}\mathcal{L}_p[\mathcal{F}(\tau)] - \frac{1}{\mathcal{S}^q}\mathcal{L}_p[\mathcal{M}\mathcal{Y}(\tau) + \mathcal{N}\mathcal{Y}(\tau)]. \tag{8}$$

Operating the inverse p-Laplace transform, we obtain

$$\mathcal{Y}(\tau) = \mathcal{H}(\tau) - \mathcal{L}_p^{-1}\left[\frac{1}{\mathcal{S}^q}\mathcal{L}_p[\mathcal{M}\mathcal{Y}(\tau) + \mathcal{N}\mathcal{Y}(\tau)]\right], \tag{9}$$

where

$$\mathcal{H}(\tau) = \mathcal{L}_p^{-1}\left[\frac{\mathcal{G}(\tau)}{\mathcal{S}} + \frac{1}{\mathcal{S}^q}\mathcal{L}_p[\mathcal{F}(\tau)]\right]. \tag{10}$$

The method represents the solution as an infinite series

$$\mathcal{Y}(\tau) = \sum_{n=0}^\infty \mathcal{Y}_n(\tau), \tag{11}$$

and the term $\mathcal{N}\mathcal{Y}(\tau)$ is given by

$$\mathcal{N}\mathcal{Y}(\tau) = \sum_{n=0}^\infty \mathcal{A}_n, \tag{12}$$

where the A domain polynomials $\mathcal{A}_n$ can be formed as

$$\mathcal{A}_n = \frac{1}{n!}\frac{d^n}{d\mathfrak{I}^n}\left[\mathcal{N}\sum_{i=0}^n \mathfrak{I}^i\mathcal{Y}_i\right]_{\beta=0}. \tag{13}$$

Substituting Equations (12) and (13) into Equation (9) yields

$$\sum_{n=0}^\infty \mathcal{Y}_n(\tau) = \mathcal{H}(\tau) - \mathcal{L}_p^{-1}\left[\frac{1}{\mathcal{S}^q}\mathcal{L}_p\left[\mathcal{M}\sum_{n=0}^\infty \mathcal{Y}_n(\tau) + \mathcal{A}_n\right]\right]. \tag{14}$$

We obtain the recursive relation

$$\mathcal{Y}_0(\tau) = \mathcal{H}(\tau),$$

$$\mathcal{Y}_{n+1}(\tau) = -\mathcal{L}_p^{-1}\left[\frac{1}{\mathcal{S}^q}\mathcal{L}_p[\mathcal{M}\mathcal{Y}_n(\tau) + \mathcal{A}_n]\right], \quad n \geq 0. \tag{15}$$

The approximate solution can be expressed as

$$\mathscr{Y}(\tau) = \sum_{n=0}^{\infty} \mathscr{Y}_n(\tau). \tag{16}$$

4. Convergence Analysis

In this section, we established the convergence and uniqueness of the p-Laplace decomposition method.

Theorem 1. *The p-Laplace decomposition solution of (6) is unique whenever $0 < (\mathscr{C}_1 + \mathscr{C}_2)(\tau^q/\Gamma[1+q]) < 1$.*

Theorem 2. *The p-Laplace decomposition method solution of the problem (6) is convergent.*

Proof. Let $\mathscr{Y}_\iota = \sum_{\nu=0}^{\iota} \mathscr{Y}_\nu(\tau)$. To prove that $\mathscr{Y}_\iota$ is a Cauchy sequence in Banach space W, we consider

$$
\begin{aligned}
\|\mathscr{Y}_\iota - \mathscr{Y}_\kappa\| &= \max_{\tau \in \mathscr{T}} |\mathscr{Y}_\iota - \mathscr{Y}_\kappa| \\
&= \max_{\tau \in \mathscr{T}} \left| \sum_{\nu=\kappa+1}^{\iota} \mathscr{Y}_\nu(\tau) \right| \\
&\leq \max_{\tau \in \mathscr{T}} \left| \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p \left[\sum_{\nu=\kappa+1}^{\iota} (\mathscr{M}(\mathscr{Y}_{\nu-1}(\tau)) + \mathscr{N}(\mathscr{Y}_{\nu-1}(\tau))) \right] \right] \right| \\
&= \max_{\tau \in \mathscr{T}} \left| \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p \left[\sum_{\nu=\kappa}^{\iota-1} (\mathscr{M}(\mathscr{Y}_\nu(\tau)) + \mathscr{N}(\mathscr{Y}_\nu(\tau))) \right] \right] \right| \\
&\leq \max_{\tau \in \mathscr{T}} \left| \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p[(\mathscr{M}(\mathscr{Y}_{\iota-1}) - \mathscr{M}(\mathscr{Y}_{\kappa-1}) \right. \right. \\
&\quad \left. \left. + \mathscr{N}(\mathscr{Y}_{\iota-1}) - \mathscr{N}(\mathscr{Y}_{\kappa-1}))] \right] \right| \\
&\leq \mathscr{C}_1 \max_{\tau \in \mathscr{T}} \left| \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p[(\mathscr{Y}_{\iota-1} - \mathscr{Y}_{\kappa-1})] \right] \right| \\
&\quad + \mathscr{C}_2 \max_{\tau \in \mathscr{T}} \left| \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p[(\mathscr{Y}_{\iota-1} - \mathscr{Y}_{\kappa-1})] \right] \right| \\
&= (\mathscr{C}_1 + \mathscr{C}_2) \frac{\tau^q}{\Gamma[1+q]} \|\mathscr{Y}_{\iota-1} - \mathscr{Y}_{\kappa-1}\|.
\end{aligned}
\tag{17}
$$

Let $\iota = \kappa + 1$, we have

$$\|\mathscr{Y}_{\kappa+1} - \mathscr{Y}_\kappa\| \leq \mathscr{C} \|\mathscr{Y}_\kappa - \mathscr{Y}_{\kappa-1}\| \leq \mathscr{C}^2 \|\mathscr{Y}_{\kappa-1} - \mathscr{Y}_{\kappa-2}\| \leq \cdots \leq \mathscr{C}^\kappa \|\mathscr{Y}_1 - \mathscr{Y}_0\|, \tag{18}$$

where $\mathscr{C} = (\mathscr{C}_1 + \mathscr{C}_2)(\tau^q/\Gamma[1+q])$. In the same vein,

$$
\begin{aligned}
\|\mathscr{Y}_\iota - \mathscr{Y}_\kappa\| &\leq \|\mathscr{Y}_{\kappa+1} - \mathscr{Y}_\kappa\| + \|\mathscr{Y}_{\kappa+2} - \mathscr{Y}_{\kappa+1}\| + \cdots + \|\mathscr{Y}_\iota - \mathscr{Y}_{\iota-1}\| \\
&\leq \left(\mathscr{C}^\kappa + \mathscr{C}^{\kappa+1} + \cdots \mathscr{C}^{\iota-1} \right) \|\mathscr{Y}_1 - \mathscr{Y}_0\| \\
&\leq \mathscr{C}^\kappa \left(\frac{1 - \mathscr{C}^{\iota-\kappa}}{1 - \mathscr{C}} \right) \|\mathscr{Y}_1\|.
\end{aligned}
\tag{19}
$$

Since $0 < \mathscr{C} < 1$, we get $1 - \mathscr{C}^{\iota-\kappa} < 1$.

$$\|\mathscr{Y}_\iota - \mathscr{Y}_\kappa\| \leq \frac{\mathscr{C}^\kappa}{1 - \mathscr{C}} \max_{\tau \in \mathscr{T}} \|\mathscr{Y}_1\|. \tag{20}$$

Since $\|\mathscr{Y}_1\| < \infty$, $\|\mathscr{Y}_\iota - \mathscr{Y}_\kappa\| \longrightarrow 0$ as $\kappa \longrightarrow \infty$; hence, $\mathscr{Y}_\iota$ is a Cauchy sequence, and then, the series is convergent. $\square$

5. Test Examples

Example 1. We first start with the Riccati equation [24]:

$$\mathfrak{D}_0^{q,p} \mathscr{Y}(\tau) = 2\mathscr{Y}(\tau) - \mathscr{Y}^2(\tau) + 1, \quad 0 < q \leq 1, \tau > 0, \tag{21}$$

subject to $\mathscr{Y}(0) = 0$.

Take the p-Laplace transform, we obtain

$$\mathscr{L}_p[\mathscr{Y}(\tau)] = \frac{1}{s^{q+1}} + \frac{1}{s^q} \mathscr{L}_p[2\mathscr{Y}(\tau) - \mathscr{Y}^2(\tau)]. \tag{22}$$

Applying the inverse p-Laplace transform, we have

$$\mathscr{Y}(\tau) = \frac{\tau^{qp}}{p^q \Gamma[q+1]} + \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p[2\mathscr{Y}(\tau) - \mathscr{Y}^2(\tau)] \right]. \tag{23}$$

Therefore,

$$\sum_{n=0}^{\infty} \mathscr{Y}_n(\tau) = \frac{\tau^{qp}}{p^q \Gamma[q+1]} + \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \mathscr{L}_p \left[2 \sum_{n=0}^{\infty} \mathscr{Y}_n(\tau) - \sum_{n=0}^{\infty} \mathscr{N}_n \right] \right], \tag{24}$$

where the nonlinear term $\mathscr{Y}^2(\tau) = \sum_{n=0}^{\infty} \mathscr{N}_n$.
We define the following recursively formula:

$$
\begin{aligned}
\mathscr{Y}_0(\tau) &= \frac{\tau^{qp}}{p^q \Gamma[q+1]}, \\
\mathscr{Y}_{n+1}(\tau) &= \mathscr{L}_p^{-1} \left[\frac{1}{s^q} \ell_q[2\mathscr{Y}_n(\tau) - \mathscr{N}_n] \right], \quad n \geq 0.
\end{aligned}
\tag{25}
$$

This gives

$$\mathscr{Y}_0(\tau) = \frac{\tau^{qp}}{p^q \Gamma[1+q]},$$

$$\mathscr{Y}_1(\tau) = \frac{2\tau^{2qp}}{p^{2q} \Gamma[1+2q]} - \frac{\Gamma[1+2q]\tau^{3qp}}{p^{3q} \Gamma^2[1+q] \Gamma[1+3q]},$$

$$
\begin{aligned}
\mathscr{Y}_2(\tau) &= \frac{4\tau^{3qp}}{p^{3q} \Gamma[1+3q]} - \frac{2\Gamma[1+2q]\tau^{4qp}}{p^{4q} \Gamma^2[1+q] \Gamma[1+4q]} \\
&\quad - \frac{4\Gamma[1+3q]\tau^{4qp}}{p^{4q} \Gamma[1+q] \Gamma[1+2q] \Gamma[1+4q]} \\
&\quad + \frac{2\Gamma[1+2q] \Gamma[1+4q]\tau^{5qp}}{p^{5q} \Gamma^3[1+q] \Gamma[1+3q] \Gamma[1+5q]},
\end{aligned}
$$

$$\vdots \tag{26}$$

Therefore, the series solution is given by

$$
\begin{aligned}
\mathcal{Y}(\tau) = {} & \frac{\tau^{q\rho}}{\rho^{q}\Gamma[1+q]} + \frac{2\tau^{2q\rho}}{\rho^{2q}\Gamma[1+2q]} \\
& - \frac{\Gamma[1+2q]\tau^{3q\rho}}{\rho^{3q}\Gamma^{2}[1+q]\Gamma[1+3q]} + \frac{4\tau^{3q\rho}}{\rho^{3q}\Gamma[1+3q]} \\
& - \frac{2\Gamma[1+2q]\tau^{4q\rho}}{\rho^{4q}\Gamma^{2}[1+q]\Gamma[1+4q]} - \frac{4\Gamma[1+3q]\tau^{4q\rho}}{\rho^{4q}\Gamma[1+q]\Gamma[1+2q]\Gamma[1+4q]} \\
& + \frac{2\Gamma[1+2q]\Gamma[1+4q]\tau^{5q\rho}}{\rho^{5q}\Gamma^{3}[1+q]\Gamma[1+3q]\Gamma[1+5q]} + \cdots .
\end{aligned}
$$
(27)

The solution is given by Equation (27), when $\rho = 1$, $q = 1$ is similar to the exact solution provided by

$$
\mathcal{Y}(\tau) = 1 + \sqrt{2}\tanh\left[\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right].
$$
(28)

In Table 1, we exhibit the numerical solutions of Equation (21), obtained by the ρ-Laplace decomposition method when $\rho = 1$, $q = 1$. Table 2 displays the numerical results obtained by the ρ-Laplace decomposition method to the fractional Riccati Equation (21) for several values of ρ and q. We plot the approximate solution and exact solution of Riccati Equation (21) in Figure 1. In Figure 2, we plot the approximate solutions to Equation (21) with different values of ρ and q. From Table 1 and Figure 1, we can deduce that the $\mathcal{Y}$-approximate solution of Equation (21) is nearly identical to the $\mathcal{Y}$-exact solution (28). Table 2 and Figure 2 show that for different values of ρ and q the solutions have the same behavior.

Example 2. In the last problem, we discuss the fractional Chen system [24]:

$$
\mathfrak{D}_{0}^{q,\rho}x(\tau) = d(y(\tau) - x(\tau)),
$$
(29)

$$
\mathfrak{D}_{0}^{q,\rho}y(\tau) = (f - d)x(\tau) - x(\tau)z(\tau) + fy(\tau),
$$
(30)

$$
\mathfrak{D}_{0}^{q,\rho}z(\tau) = x(\tau)y(\tau) - ez(\tau),
$$
(31)

subject to $x(0) = a$, $y(0) = b$, and $z(0) = c$, where $d, e, f \in \mathbb{R}$, $t > 0$, $0 < q \le 1$.

Solution. Applying the ρ-Laplace transform to Equation (29) and substituting the initial conditions, we have

$$
\mathcal{L}_{\rho}[x(\tau)] = \frac{a}{\jmath} + \frac{d}{\jmath^{q}}\mathcal{L}_{\rho}[y(\tau) - x(\tau)],
$$
(32)

$$
\mathcal{L}_{\rho}[y(\tau)] = \frac{b}{\jmath} + \frac{1}{\jmath^{q}}\mathcal{L}_{\rho}[(f-d)x(\tau) - x(\tau)z(\tau) + fy(\tau)],
$$
(33)

TABLE 1: Numerical results of the Riccati equation in Example 1.

τ	$\mathcal{Y}$-approximate	$\mathcal{Y}$-exact	Absolute error
0.1	0.1102680000	0.1102951969	0.00002719692
0.2	0.2416426667	0.2419767996	0.00033413295
0.3	0.393924	0.3951048487	0.00118084866
0.4	0.565632	0.5678121663	0.00218016629
0.5	0.7541666667	0.7560143934	0.00184772676

TABLE 2: Numerical results of the Riccati equation for various values of ρ and q.

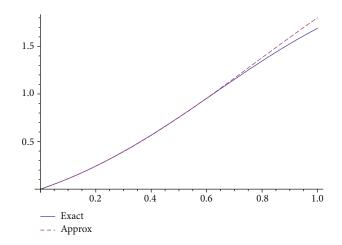| τ	$\rho = 0.9, q = 1$	$\rho = 0.75, q = 0.95$	$\rho = 1.2, q = 0.9$
0.2	0.3321562273	0.6242722905	0.1829684851
0.4	0.7290020600	1.1880854973	0.4535900161
0.6	1.1701070672	1.6946267332	0.8013841691
0.8	1.6131205961	2.1013183873	1.2035596201
1	2.0126222854	2.3786954138	1.6223167380

FIGURE 1: A comparison of the exact solution and three terms' approximate solution when $\rho = 1$, $q = 1$.

$$
\mathcal{L}_{\rho}[z(\tau)] = \frac{c}{\jmath} + \frac{1}{\jmath^{q}}\mathcal{L}_{\rho}[x(\tau)y(\tau) - ez(\tau)].
$$
(34)

Taking $\mathcal{L}_{\rho}^{-1}$ to Equation (32), we get

$$
x(\tau) = a + \mathcal{L}_{\rho}^{-1}\left[\frac{d}{\jmath^{q}}\mathcal{L}_{\rho}[y(\tau) - x(\tau)]\right],
$$

$$
y(\tau) = b + \mathcal{L}_{\rho}^{-1}\left[\frac{1}{\jmath^{q}}\mathcal{L}_{\rho}[(f-d)x(\tau) - x(\tau)z(\tau) + fy(\tau)]\right],
$$

$$
z(\tau) = c + \mathcal{L}_{\rho}^{-1}\left[\frac{1}{\jmath^{q}}\mathcal{L}_{\rho}[x(\tau)y(\tau) - ez(\tau)]\right].
$$
(35)

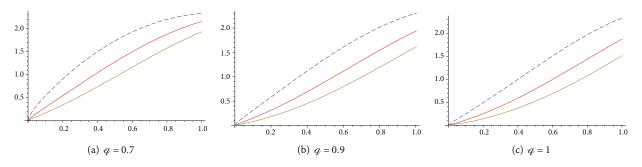(a) $q = 0.7$ (b) $q = 0.9$ (c) $q = 1$

FIGURE 2: Plots of approximate solutions for Example 1, when $p = 0.8$ (dashed), $p = 1$ (red), and $p = 1.2$ (thick).

Therefore,

$$\sum_{n=0}^{\infty} x_n(\tau) = a + \mathscr{L}_p^{-1}\left[\frac{d}{s^q}\,\mathscr{L}_p\left[\sum_{n=0}^{\infty} y_n(\tau) - \sum_{n=0}^{\infty} x_n\right]\right],$$

$$\sum_{n=0}^{\infty} y_n(\tau) = b + \mathscr{L}_p^{-1}\left[\frac{1}{s^q}\,\mathscr{L}_p\left[(f-d)\sum_{n=0}^{\infty} x_n(\tau) - \sum_{n=0}^{\infty} A_n\right] + f\sum_{n=0}^{\infty} y_n(\tau)\right],$$

$$\sum_{n=0}^{\infty} z_n(\tau) = c + \mathscr{L}_p^{-1}\left[\frac{1}{s^q}\,\mathscr{L}_p\left[\sum_{n=0}^{\infty} B_n(\tau) - e\sum_{n=0}^{\infty} z_n\right]\right].$$

(36)

The nonlinear terms are given by $x(\tau)z(\tau) = \sum_{n=0}^{\infty} A_n$ and $x(\tau)y(\tau) = \sum_{n=0}^{\infty} B_n$.

The terms of the solution are given by

$$x_0(\tau) = a, \tag{37}$$

$$y_0(\tau) = b, \tag{38}$$

$$z_0(\tau) = c, \tag{39}$$

$$x_{n+1}(\tau) = a + \mathscr{L}_p^{-1}\left[\frac{d}{s^q}\,\mathscr{L}_p[y_n(\tau) - x_n]\right], \quad n \geq 0, \tag{40}$$

$$y_{n+1}(\tau) = b + \mathscr{L}_p^{-1}\left[\frac{1}{s^q}\,\mathscr{L}_p[(f-d)x_n(\tau) - A_n] + f y_n(\tau)\right], \quad n \geq 0, \tag{41}$$

$$z_{n+1}(\tau) = c + \mathscr{L}_p^{-1}\left[\frac{1}{s^q}\,\mathscr{L}_p[B_n(\tau) - e z_n]\right], \quad n \geq 0. \tag{42}$$

This gives

$$x_0(\tau) = a,$$
$$y_0(\tau) = b,$$
$$z_0(\tau) = c,$$
$$x_1(\tau) = \frac{d\tau^{qp}[b-a]}{p^q\Gamma[q+1]},$$
$$y_1(\tau) = \frac{\tau^{qp}[(f-d)a - ac + fb]}{p^q\Gamma[q+1]},$$
$$z_1(\tau) = \frac{\tau^{qp}[ab - ce]}{p^q\Gamma[q+1]}, \tag{43}$$
$$x_2(\tau) = \frac{d\tau^{2qp}[fa - ac - (d-f)b]}{p^{2q}\Gamma[2q+1]},$$
$$y_2(\tau) = \frac{\tau^{2qp}\left[-a^2b + f^2(a+b) - bd(c+d) + a(ec + d(c+d)) + f(bd - a(c+2d))\right]}{p^{2q}\Gamma[2q+1]},$$
$$z_2(\tau) = \frac{\tau^{2qp}[db(-a+b) + e(-ab + ec) + a(-da + f(a+b) - ac)]}{p^{2q}\Gamma[q+1]},$$
$$\vdots$$

TABLE 3: Numerical outcome of Equation (29) for $p = 1$, $q = 1$, $(a, b, c) = (-10, 0, 37)$, and $(d, e, f) = (35, 3, 28)$.

τ	$x(\tau)$	$y(\tau)$	$z(\tau)$
1	$-2.7674260625 \times 10^7$	$9.980182137223611 \times 10^9$	$4.23914339125 \times 10^7$
2	$-1.890854523333333 \times 10^9$	$3.184114879068444 \times 10^{11}$	$2.976572864333333 \times 10^9$
3	$-2.1994001685625 \times 10^{10}$	$2.410737003809362 \times 10^{12}$	$3.489912176376249 \times 10^{10}$
4	-124856845410	$1.012853389759111 \times 10^{13}$	$1.988762323242 \times 10^{11}$
5	$-4.792186801089583 \times 10^{11}$	$3.081744950221476 \times 10^{13}$	$7.650373367356459 \times 10^{11}$

Therefore, the series solution is as follows:

$$x(\tau) = a + \frac{d\tau^{qp}[b - a]}{p^q \Gamma[q + 1]} + \frac{d\tau^{2qp}[fa - ac - (d - f)b]}{p^{2q} \Gamma[2q + 1]} + \cdots,$$

$$y(\tau) = b + \frac{\tau^{qp}[(f - d)a - ac + fb]}{p^q \Gamma[q + 1]} + \frac{\tau^{2qp}[-a^2b + f^2(a + b) - bd(c + d) + a(ec + d(c + d)) + f(bd - a(c + 2d))]}{p^{2q} \Gamma[2q + 1]} + \cdots,$$

$$z(\tau) = c + \frac{\tau^{qp}[ab - ce]}{p^q \Gamma[q + 1]} + \frac{\tau^{2qp}[db(-a + b) + e(-ab + ec) + a(-da + f(a + b) - ac)]}{p^{2q} \Gamma[q + 1]} + \cdots.$$

$$(44)$$

We have displayed 7-term solution points of the Chen system (29) for $p = 1$, $q = 1$, $(a, b, c) = (-10, 0, 37)$, and $(d, e, f) = (35, 3, 28)$. The numerical outcomes obtained are presented in Table 3.

6. Conclusion

In this research, we effectively used the p-Laplace decomposition approach to provide an approximate solution for initial value problems with generalized fractional derivatives. We tested the strategy in two different situations. The results revealed that the technique is exceedingly successful with a small number of calculations and is devoid of any linearization, perturbations, discretization, or restrictive assumptions. In addition, it is established that the findings obtained in the series form have a higher rate of convergence to the exact results. This approach can be used to solve various fractional PDEs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflict of interest.

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