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Research Article

Corrected Inertial Torques of Gyroscopic Effects

Ryspek Usubamatov[1] and Dennis Allen[2]

[1]*Kyrgyz State Technical University after I, Razzakov, Bishkek, Kyrgyzstan*
[2]*17046 Lloyds Bayou Dr. Apt 322, Spring Lake, MI 49456, USA*

Correspondence should be addressed to Ryspek Usubamatov; ryspek0701@yahoo.com

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The published manuscripts in the area of gyroscope theory were presented mainly by the simplified approaches in which mathematical models contain many uncertainties. New research in machine dynamics opened breakthrough directions in gyroscopic effects of rotating objects that give the correct solutions. The pioneering work meets many problems when solving the scientific innovations that are accompanied by successes and omissions. New mathematical models for the gyroscopic inertial torques were derived with incorrect processing of the integral equations that give distorted results. The gyroscopic devices in engineering manifest gyroscopic effects as the action of the inertial torques which computing is crucial for mathematical describing of their motions. The corrected mathematical processing of the equations for the inertial torques acting in a gyroscope is presented in this manuscript.

1. Introduction

The dynamics of rotating objects is the most complex part of engineering mechanics. Among the problems of the machine dynamics, the gyroscopic effects stand out long time by their being unsolved [1–4]. Physicists and mathematicians of the different generations developed only partial analytical solutions that do not solve the entire problem. The numerical models for gyroscopic effects do not describe their physics [5–11]. The unsolved nature of gyroscopic effects attracts new researchers, which today publish many works of the study gyroscopic problems [12, 13].

New investigations in the gyroscope theory showed the physics of the gyroscopic effects and opened a new direction in the dynamics of the mechanics [14–16]. The mathematical models for the gyroscopic effects show their complexity in analytical processing based on several physical principles. The physical laws of classical mechanics that were discovered at different times formulated the gyroscope theory and some of them were unknown to the scientist of the past centuries. The scientists of our time did not use the basic physical principles and methods for the formulation of the gyroscope theory. The new study of gyroscopic effects shows the action of the system of the inertial torques about axes of rotations. The gyroscopic motions are kinetically interrelated by the principle of the energy conservation law. The published mathematical models for inertial torques generated by the rotating mass of the spinning disc contain an error in mathematical processing and an error in the title [14, 16]. The incorrect expression of the inertial torque generated by the centrifugal forces of the spinning disc is $T_{ct.x} = (2\pi^2/9)J\omega\omega_x$, that the real value is twice bigger. The title of "common inertial forces" is not correct because, in reality, it is the centrifugal forces acting around another axis. The errors are mechanical and not principle.

All gyroscopic devices and rotating objects in engineering manifest gyroscopic effects as the action of the inertial torques, which accurate computing is crucial for mathematical describing of their motions. The exact mathematical models for gyroscopic inertial torques enable calculating the angular velocities of the gyroscope around axes of rotations. The known publications describe the gyroscope motions by the action only of the precession torque of the change in the angular momentum $T_{am.x} = J\omega\omega_x$ [1–8]. This torque is one component of the system of inertial torque [14, 16] acting on gyroscopic devices and does not explain other gyroscopic effects.

The manuscript contributions are presented by the corrected inertial torques of the centrifugal forces acting around two axes. The expression "the common inertial forces and torque" is not correct and is removed from the text. The expression of the inertial torque generated by the Coriolis forces remained the same but corrected the limits of the integral equation for its solution. The corrected centrifugal inertial torques generated by the spinning rotor are represented practically in the mathematical modeling of gyroscope motions.

2. Methodology

The action of the external torque activates several inertial torques on the spinning disc which manifests the gyroscopic effects. The method of analytical solution for the gyroscopic inertial torques produced by the centrifugal, Coriolis forces, and the change in the angular momentum of the spinning disc is presented in publications [14–16]. The inertial torques of the centrifugal and the Coriolis forces are generated by the distributed masses of the disc and defined by the integral equations. Their mathematical models for the inertial torques were obtained by the incorrect presentations of the integral limits. Additionally, the centrifugal forces act around two axes and one force was called the common inertial force erroneously. The errors in the integral limits yield the change in the expressions for the inertial torques that are fundamental in gyroscope theory.

Analysis of the inertial torques, described in the manuscript [14–16], shows their nature are the centrifugal and Coriolis forces acting about two axes when the spinning disc rotates about three axes. The action of these forces about one axis produces the resistance and the precession torques about two axes. The method for the solution for the inertial torques generated by the centrifugal and Coriolis forces of a spinning disc is described in detail in the publications [14–16]. The corrected symbols of the inertial torques acting on the spinning disc are presented in Figure 1.

Figure 1 shows the action of the torques and motions of the spinning disc around axes. Where T is the external torque; $T_{ct\cdot i}$,$T_{cr\cdot i}$, and $T_{am\cdot i}$ are the inertial torques generated by the centrifugal, Coriolis forces and the change in the angular momentum acting about axis i, respectively; ω, ω_x, and ω_y are the angular velocity of the disc about axis *oz, ox,* and *oy,* respectively; R is the radius of the disc, and γ is the angle of the inclination of the disc axle. The action of the torques and motions of the disc are accepted as positive in the counterclockwise and negative in the clockwise directions, respectively. These signs are presented in the subsequent analytical expressions.

2.1. The Inertial Torque Generated by Centrifugal Forces of Rotating Mass Elements. The centrifugal forces generated by the rotating mass elements m of the spinning disc disposed on its radius $(2/3)R$ (Figure 2). The expression of the m is $m = M\Delta\delta/2\pi$, in which M is the mass of the disc and $\Delta\delta$ is the sector's angle of the mass element's disposition. The expression of the centrifugal force of the rotating mass element is$f_{ct} = M(2/3)R\omega^2\Delta\delta/2\pi = MR\omega^2\Delta\delta/3\pi$. The centrifugal force acting along with axis *oz* has the following expressions:

(i) For the axis *ox*

$$f_{ct.x} = -\frac{M}{3\pi}R\omega^2 \sin\alpha\Delta\delta \sin\Delta\gamma = -\frac{M}{3\pi}R\omega^2 \sin\alpha\Delta\delta\Delta\gamma. \quad (1)$$

(ii) For the axis *oy*

$$f_{ct.x} = -\frac{M}{3\pi}R\omega^2 \cos\alpha\Delta\delta \sin\Delta\gamma = -\frac{M}{3\pi}R\omega^2 \cos\alpha\Delta\delta\Delta\gamma. \quad (2)$$

where ω is the constant angular velocity of the disc; α is the angle of the mass element's disposition; $\Delta\gamma$ is the angle of turn for the disc's plane ($\sin\Delta\gamma = \Delta\gamma$ for the small values of the angle).

The centrifugal forces (Equations (1) and (2)) acting around axes *ox* and *oy* are the same but change by the sine and cosine laws. The expression of the inertial torque of the centrifugal force generated by the mass element of the spinning disc is

$$\Delta T_{ct.} = f_{ct.x}y_m, \quad (3)$$

where $y_m = (2/3)R\sin\alpha$ and $x_m = (2/3)R\cos\alpha$ is the distance of the mass element's disposition relative to axes *ox* and *oy*, respectively.

A distributed inertial torques, generated by the rotating mass element, produce the concentrated load applied at the centroids that are point A of Figure 2(a) and point B of Figure 2(b), which is calculated by the known integrated equation [1–3].

The steps of the mathematical modeling for the inertial torques of the centrifugal forces are the same as presented in the publications [14–16]. The comments on the mathematical processing are omitted from consideration but are given explanations.

The integrated components of the centrifugal forces act on the centroids which expressions are

(i) For the axis *ox*

$$\begin{aligned} y_A &= \frac{\int_{\alpha=0}^{\pi} f_{ct.z} y_m d\alpha}{\int_{\alpha=0}^{\pi} f_{ct.z} d\alpha} \\ &= \frac{\int_{\alpha=0}^{\pi}(M/3\pi)R\omega^2\Delta\delta\Delta\gamma \sin\alpha (2/3)R\sin\alpha d\alpha}{\int_{\alpha=0}^{\pi}(M/3\pi)R\omega^2\Delta\delta\Delta\gamma\sin\alpha d\alpha} \\ &= \frac{(M/3\pi)R\omega^2\Delta\delta\Delta\gamma\int_{\alpha=0}^{\pi}(2/3)R\sin^2\alpha d\alpha}{(M/3\pi)R\omega^2\Delta\delta\Delta\gamma\int_{\alpha=0}^{\pi}\sin\alpha d\alpha} \\ &= \frac{(2/3)R\int_{\alpha=0}^{\pi}(1/2)(1-\cos 2\alpha)d\alpha}{\int_{\alpha=0}^{\pi}\sin\alpha d\alpha}. \end{aligned} \quad (4)$$

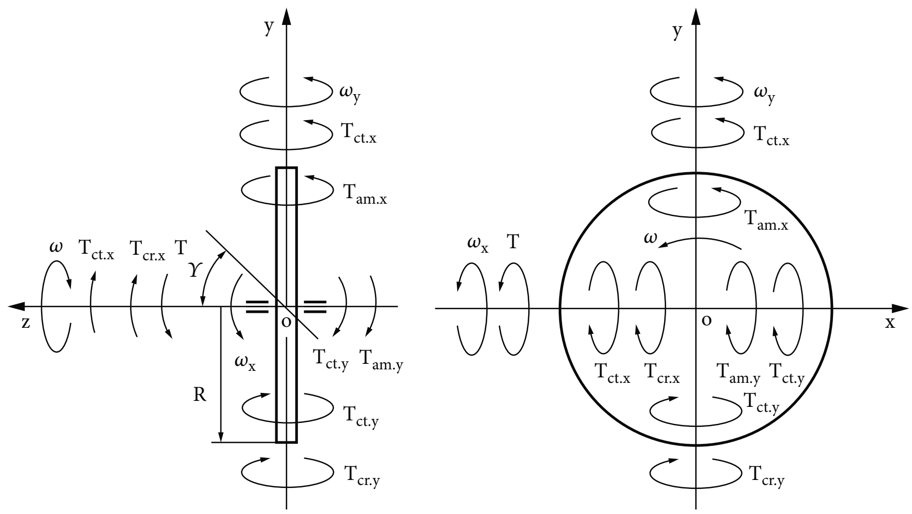

FIGURE 1: External and inertial torques act on the spinning disc.

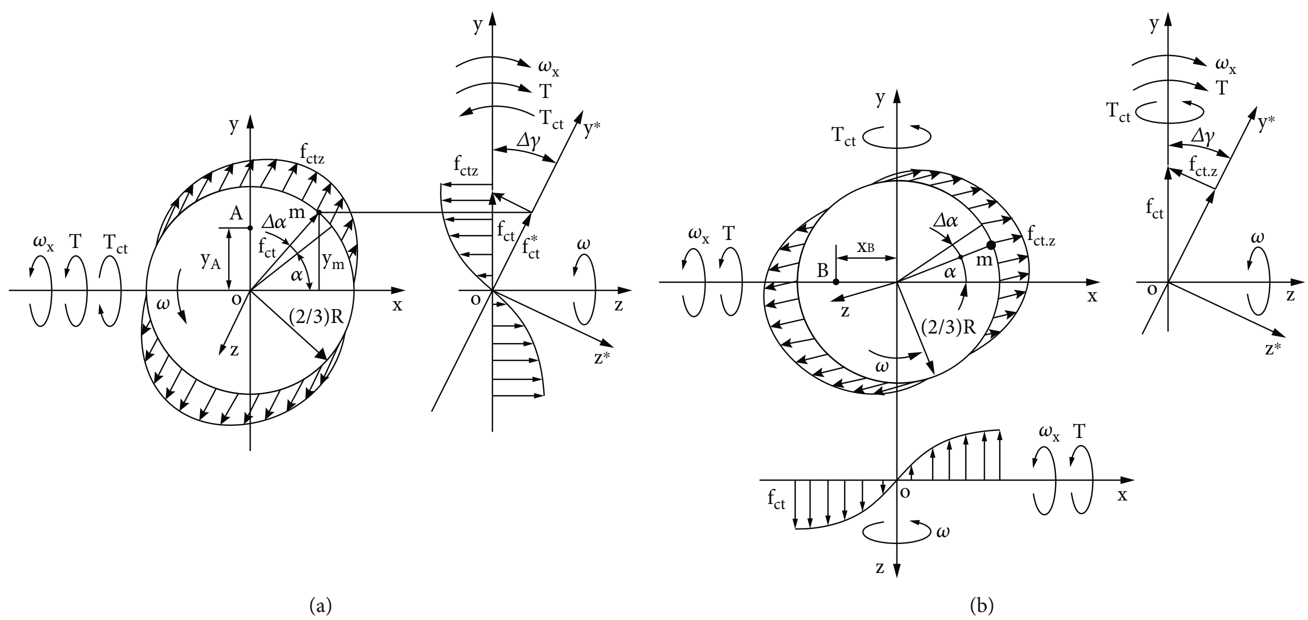

FIGURE 2: Schematic of acting centrifugal forces, torques, and motions of the rotating disc about axis *ox* (a) and axis *oy* (b).

(ii) For the axis *oy*

$$
\begin{aligned}
y_A &= \frac{\int_{\alpha=0}^{\pi} f_{ct.z} y_m d\alpha}{\int_{\alpha=0}^{\pi} f_{ct.z} d\alpha} \\
&= \frac{\int_{\alpha=0}^{\pi} (M/3\pi) R\omega^2 \Delta\delta\Delta\gamma \cos\alpha (2/3) R \cos\alpha d\alpha}{\int_{\alpha=0}^{\pi} (M/3\pi) R\omega^2 \Delta\delta\Delta\gamma \cos\alpha d\alpha} \\
&= \frac{(M/3\pi) R\omega^2 \Delta\delta\Delta\gamma \int_{\alpha=0}^{\pi} (2/3) R \cos^2\alpha d\alpha}{(M/3\pi) R\omega^2 \Delta\delta\Delta\gamma \int_{\alpha=0}^{\pi} \cos\alpha d\alpha} \\
&= \frac{(2/3) R \int_{\alpha=0}^{\pi} (1/2)(1+\cos 2\alpha) d\alpha}{\int_{\alpha=0}^{\pi} \cos\alpha d\alpha} \\
&= \frac{(R/3) \int_{\alpha=0}^{\pi} (1+\cos 2\alpha) d\alpha}{\int_{\alpha=0}^{\pi} \cos\alpha d\alpha} ,
\end{aligned} \tag{5}
$$

where the expression $f_{ct.z}$ is accepted as constant for Equations (4) and (5), the expressions$\sin^2\alpha = (1\text{-}\cos 2\alpha)/2$ and $\cos^2\alpha = (1+\cos 2\alpha)/2$ trigonometric identities that are replaced in the equations.

The differential forms of integral equations for the inertial torque generated by the centrifugal forces (Equation (3)) are presented by the integral forms:

(i) For the axis *ox*

$$
\begin{aligned}
\int_0^{T_{ct}} dT_{ct} = -\frac{MR\omega^2}{3\pi} \times \int_0^{\pi} d\delta \times \int_0^{\gamma} d\gamma \times \int_0^{\pi} \cos\alpha d\alpha \\
\times \frac{R\int_0^{\pi} (1-\cos 2\alpha) d\alpha}{3\int_0^{\pi} \sin\alpha d\alpha} .
\end{aligned} \tag{6}
$$

(ii) For the axis *oy*

$$\int_0^{T_{ct}} dT_{ct} = -\frac{MR\omega^2}{3\pi} \times \int_0^{\pi} d\delta \times \int_0^{\gamma} d\gamma \times \int_0^{\pi} - \sin \alpha da \frac{R\int_0^{\pi}(1+\cos 2\alpha)d\alpha}{3\int_0^{\pi} \sin \alpha d\alpha}. \quad (7)$$

The solutions of integrals Equation (6) and (7) are as follows:

(iii) For the axis *ox*

$$T_{ct}\Big|_0^{T_{ct}} = -\frac{MR\omega^2}{3\pi} \times \left(\delta|_0^{\pi}\right) \times \left(\gamma|_0^{\gamma}\right) \times \left(2 \sin \alpha|_0^{\pi/2}\right) \frac{R[\alpha - (1/2) \sin 2\alpha]|_0^{\pi}}{-3 \cos \alpha|_0^{\pi}}. \quad (8)$$

(iv) For the axis *oy*

$$T_{ct}\Big|_0^{T_{ct}} = -\frac{MR\omega^2}{3\pi} \times \left(\delta|_0^{\pi}\right) \times \left(\gamma|_0^{\gamma}\right) \times \left(\cos \alpha|_0^{\pi}\right) \frac{R[\alpha + (1/2) \sin 2\alpha]|_0^{\pi}}{-3 \cos \alpha|_0^{\pi}}, \quad (9)$$

thus giving rise to the following:

(v) For the axis *ox*

$$T_{ct} = -\frac{MR\omega^2}{3\pi} \times (\pi - 0) \times (\gamma - 0) \times (2)(1-0) \times \frac{R(\pi - 0)}{-3(-1-1)} = -\frac{MR^2\omega^2\pi}{9} \times \gamma. \quad (10)$$

(vi) For the axis *oy*

$$T_{ct} = -\frac{MR\omega^2}{3\pi} \times (\pi - 0) \times (\gamma - 0) \times (-1-1) \times \frac{R(\pi - 0)}{3 \times 2(1-0)} = \frac{MR^2\omega^2\pi}{9} \times \gamma. \quad (11)$$

Equations (10) and (11) are identical and contain the variable angle γ that depends on the angular velocity ω_x. One expression of the inertial torques T_{ct} with signs (+) and (-) is used for the subsequent solutions.

The angle γ is variable and should be expressed by the angular velocity of the disc about axis *ox*. The rate of change in torque T_{ct} per time is expressed by the differential equation.

$$\frac{dT_{ct}}{dt} = \pm\frac{MR^2\omega^2\pi}{9}\frac{d\gamma}{dt}, \quad (12)$$

where $t = \alpha/\omega$ is the time taken relative to the angular velocity of the disc, other parameters are as specified above.

The differential of time is $dt = d\alpha/\omega$; the expression $d\gamma/dt = \omega_x$ is the angular velocity of the spinning disc about axis *ox*. Substituting the defined components into Equation (12) and transformation yield:

$$\frac{\omega dT_{ct}}{d\alpha} = \pm\frac{MR^2\omega^2\omega_x\pi}{9}. \quad (13)$$

Separation of the variables of Equation (13), transformation, and presentation in the integral form with defined limits yields

$$\int_0^{T_{ct}} dT_{ct} = \pm\int_0^{\pi}\frac{MR^2\omega\omega_x\pi}{9}d\alpha. \quad (14)$$

Solving Equation (14) yields

$$T_{ct}\Big|_0^{T_{ct}} = \pm\frac{MR^2\omega\omega_x\pi}{9}\alpha|_0^{\pi}, \quad (15)$$

giving rise to the following:

$$T_{ct} = \pm\frac{MR^2\omega\omega_x\pi}{9}(\pi - 0) = \pm\frac{MR^2\pi^2\omega\omega_x}{9}. \quad (16)$$

The torque generated by the centrifugal force acts on the upper and lower and left and right sides of the plane of the disc; then, the result of Equation (16) is multiplied by two.

$$T_{ct} = \pm\frac{2 \times 2 \times \pi^2 MR^2\omega\omega_x}{9 \times 2} = \pm\frac{4}{9}\pi^2 J\omega\omega_x, \quad (17)$$

where $J = MR^2/2$ is the disc's moment of inertia, the signs (-) and (+) are for the resistance and precession torques acting about axis *ox* and *oy*, respectively, other parameters are as specified above.

The precession inertial torque acting about axing *oy* is generated by the centrifugal force that is not the common inertial force, which title is not correct. The expression of the inertial torque (Equation (17)) is different from the known publications [14, 16].

2.2. The Inertial Torque Generated by the Coriolis Forces of Rotating Mass Elements. The change in the tangential velocity of the mass elements along the disc rotating about the axis *ox* presents the Coriolis acceleration. The mass element m (Equation (1)) rotates with the constant tangential velocity V. The change in the tangential velocity V along axis *oz* is the acceleration of the mass element a_z.

$$\Delta V = V \sin \Delta\gamma, \frac{\Delta V}{\Delta t} = V\frac{\Delta\gamma}{\Delta t}, \frac{\Delta V}{\Delta t} = a_z, \frac{\Delta\gamma}{\Delta t} = \omega_x, V = \frac{2}{3}R \cos \alpha \times \omega, a_z = \frac{2}{3}R\omega\omega_x \cos \alpha, \quad (18)$$

where a_z is the Coriolis acceleration of the mass element along with axis *oz*, and $\Delta\gamma$ is the angle of the turn of the disc's plane around axis *oy* ($\sin \Delta\gamma = \Delta\gamma$ for small values of the angle).

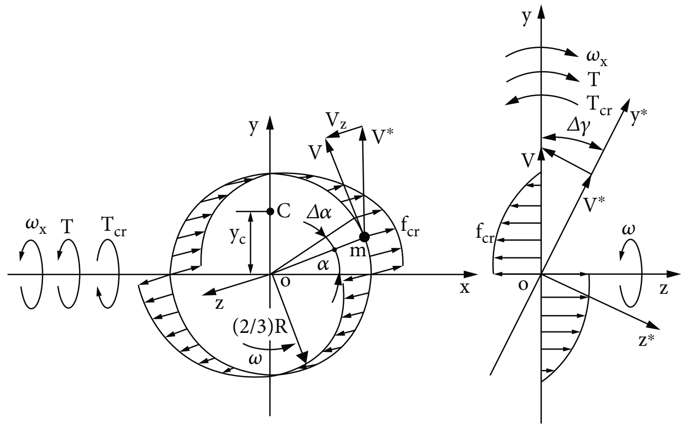

Figure 3: Schematic of acting the Coriolis forces, torques, and motions of the spinning disc about axis *ox*.

The expression for the Coriolis force of the mass element is

$$f_{cr} = -\frac{M}{2\pi} \times \Delta\delta \frac{2}{3} R\omega\omega_x \cos\alpha = -\frac{M}{3\pi} \times \Delta\delta \times R\omega\omega_x \cos\alpha, \tag{19}$$

where all parameters are as specified above.

The expression of the inertial torque for the Coriolis force generated by the mass element of the spinning disc is

$$\Delta T_{cr.} = f_{cr.z} y_m, \tag{20}$$

where $y_m = (2/3)R\sin\alpha$ is the distance of the mass element's disposition relative to axes *ox*.

The centroid for the torque ΔT_{cr} is the point C of Figure 3, which is defined by Equation (2).

$$\begin{aligned} y_C &= \frac{\int_{\alpha=0}^{\pi} f_{cr.} y_m d\alpha}{\int_{\alpha=0}^{\pi} f_{cr.} d\alpha} \\ &= \frac{\int_{\alpha=0}^{\pi} (MR\omega\omega_x/3\pi) \times \Delta\delta (2/3) R \cos\alpha \sin\alpha d\alpha}{\int_{\alpha=0}^{\pi} (MR\omega\omega_x/3\pi) \times \Delta\delta \cos\alpha d\alpha} \\ &= \frac{(MR\omega\omega_x/3\pi) \times \Delta\delta \times (2/3) R \int_0^{\pi} \sin\alpha d \sin\alpha}{(MR\omega\omega_x/3\pi) \times \Delta\delta \int_0^{\pi} \cos\alpha d\alpha} \\ &= \frac{(2/3) R \int_0^{\pi} \sin\alpha d \sin\alpha}{\int_0^{\pi} \cos\alpha d\alpha}. \end{aligned} \tag{21}$$

The differential form of integral equations for the inertial torque generated by the Coriolis forces (Equation (20)) is presented by the integral forms:

$$\begin{aligned} \int_0^{T_{cr}} dT_{cr} &= \frac{MR\omega\omega_x}{3\pi} \times \int_0^{\pi} d\delta \times \int_0^{\pi} -\sin\alpha d\alpha \\ &\quad \times \frac{(2/3) R \int_0^{\pi} \sin\alpha d \sin\alpha}{\int_0^{\pi} \cos\alpha d\alpha}. \end{aligned} \tag{22}$$

Solving integral Equation (22) yields

$$\begin{aligned} T_{cr} &= -\frac{MR\omega\omega_x}{3\pi} \times \left(\delta|_0^{\pi}\right) \times \left(\cos\alpha|_0^{\pi}\right) \\ &\quad \times \frac{(2/3) R \left(2\sin^2\alpha/2\right)|_0^{\pi/2}}{2\sin\alpha|_0^{\pi/2}}, \end{aligned} \tag{23}$$

thus giving rise to the following:

$$\begin{aligned} T_{cr} &= \frac{MR\omega\omega_x}{3\pi} \times (\pi - 0) \times (-1 - 1) \\ &\quad \times \frac{(2/3) R \times (1 - 0)}{2(1 - 0)} = -\frac{2MR^2\omega\omega_x}{9}, \end{aligned} \tag{24}$$

where all parameters are as specified above.

The inertial torque T_{cr} acts on the upper and lower sides of the disc and the result of Equation (24) is multiplied by two.

$$T_{cr} = -\frac{2 \times 2 \times 2MR^2\omega\omega_x}{9 \times 2} = -\frac{8}{9} J\omega\omega_x, \tag{25}$$

where all parameters are as specified above.

Equation (25) presents the Coriolis torque generated by the rotating mass elements of the disc. The known publications [14–16] contain the same expression as Equation (25) but there are the following differences. The mathematical processing of the Coriolis torque in the publications [14–16] does not have the consistency of the limits of the integral equations. The centroid for the Coriolis torque is accepted for the quarter of the circle that is not correct analytically. These two differences present mathematical incorrectness in publications [14–16].

The gyroscopic effects of the spinning disc are produced by the corrected system of inertial torques presented in Table 1.

2.3. Working Example. The spinning disc (Figure 1) of the moment of inertial $J = 5,0 \times 10^{-3}$ kg.m^2 is running with the angular velocity of 10 rad/s and precessing with an angular velocity of 0,003 rad/s. Determine the values of the torques generated by the centrifugal and Coriolis forces. Substituting the given data into Equations (17) and (25) and transformation yield the following results.

(1) The torque T_{ct} generated by the centrifugal forces

$$T_{ct} = \frac{4}{9}\pi^2 J\omega\omega_x = \frac{4}{9}\pi^2 \times 0,005 \times 10 \times 0,003 = 6,579 \times 10^{-4} Nm. \tag{26}$$

(2) The torque generated by Coriolis T_{cr} forces

$$T_{cr} = \left(\frac{8}{9}\right) J\omega\omega_x = \frac{8}{9} \times \times 0,005 \times 10 \times 0,003 = 1,333 \times 10^{-4} Nm. \tag{27}$$

2.4. Dependency of the Angular Velocities of the Inning Disc around Axes of Rotation. The mathematical model for the gyroscope motion is formulated by the torques acting on it and by the dependency of the angular velocities for the motions of the gyroscope about axes of the 3D coordinate system [16]. The expression of the angular velocities for the motions of the spinning disc depends on its angle γ of the inclination (Figure 1). The equality of the kinetic energies of the spinning disc is planned by the equality of the inertial torques acting about axes *ox* and *oy* [16], which corrected expression is

$$-\frac{4\pi^2}{9} J\omega\omega_x - \frac{8}{9} J\omega\omega_x - \frac{4\pi^2}{9} J\omega\omega_y - J\omega\omega_y = \frac{4\pi^2}{9} J\omega\omega_x \cos\gamma + J\omega\omega_x \cos\gamma - \frac{4\pi^2}{9} J\omega\omega_y - \frac{8}{9} J\omega\omega_y. \tag{28}$$

The right side of Equation (28) presents the resulting torque acting about axis *oy*, which value of precession torques is less on the cosγ due to the inclination of the spinning disc [15]. This torque generates the precession torques of the centrifugal force and the change in the angular momentum multiplied by cosγ. These precession torques act about axis *ox* and are presented on the left side of Equation (28). The value of $T_{p.y}$ is less on the cosγ for the same reason because all torques are projected on axes of the 3D coordinate system. Simplification of Equation (28) yields

$$\omega_y \cos\gamma = -\left[4\pi^2(1 + \cos\gamma) + 8 + 9\cos\gamma\right]\omega_x, \tag{29}$$

where the sign (-) relates to the direction of the inertial torque that can be omitted from the following analytical considerations.

Equation (29) presents the variable dependency of the angular velocities for the precession of the spinning disc for the common disposition around two axes. The angular velocity ω_y changes by the cosine law. The angular velocity ω_x depends on the value of the external load T. The disposition of the disc's axis on 90° and 270° gives the zero angular velocity of the spinning disc rotation about axis *oy*. At these dispositions, the dependency of Equation (29) does not maintain.

Table 1: Equations of the inertial torques acting on the spinning disc.

Type of the torque generated by	Action	Equation
Centrifugal forces	Resistance Precession	$T_{ct} = (4/9)\pi^2 J\omega\omega_x$
Coriolis forces	Resistance	$T_{cr} = (8/9)J\omega\omega_x$
Change in angular momentum	Precession	$T_{am} = J\omega\omega_x$

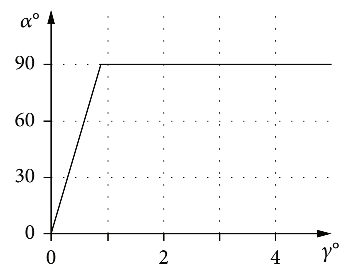

Figure 4: The angular dispositions of the outer gimbal (α) versus the angular disposition (γ) of the inner gimbal of the spinning disc.

For the angle $\gamma = 0$, the ratio ω_y/ω_x is maximal, as is follows:

$$\frac{\omega_y}{\omega_x} = \left[4\pi^2(1 + \cos 0^o) + 8 + 9\cos 0^o\right] = 8\pi^2 + 17 = 95,956835. \tag{30}$$

The turn of the spinning disc about axis *oy* on 90° is implemented by the turn of the disc about axis *ox* on the angle:

$$\gamma = \frac{90^\circ}{8\pi^2 + 17} = 0,938^\circ. \tag{31}$$

Practically, these angular motions are visible in motions of the gyroscopic gimbals. This gyroscope property is validated by the test of the gyroscope with the horizontal disposition of the spinning disc axis and the turn of the outer and inner gimbals around axes *oy* and *ox*. The defined angles of the gyroscope turn-around axes are validated by the tests on any gyroscopes. Figure 4 shows the diagram of the change in the angular disposition (γ) of the inner gimbal versus the change in the angular disposition (α) of the outer gimbal. The results have explained the physics of gyroscopic gimbal motions that was one in the series of unsolved gyroscopic effects.

3. Results and Discussion

Gyroscopic problems were remaining as sophisticated physical and mathematical tasks for a long time. The known publications dedicated to gyroscopic effects are sinned by imperfectness in mathematical modeling and attracted inquisitive researchers [1–4]. The known simplified gyroscope theories are based on the action only of the precession torque and do not describe the gyroscopic effects [5–8]. The recent publications in the area of gyroscopic effects contain the mathematical models with errors in the processing of the inertial torques generated by the centrifugal and Coriolis forces [14, 16]. Incorrect analytical processing of the expressions for the inertial torques yields their distorted expressions. Other mathematical models for the gyroscope motions have inherited errors and have given wrong results. The issues had the wrong title of the precession torque generated by the centrifugal forces of the disc rotating mass. This work presents the corrected expression for the centrifugal inertial torque acting around two axes and the corrected title for the precession torque. The obtained results should be used for mathematical modeling for gyroscopic effects and enable avoid criticism from the readers.

4. Conclusion

Researchers of all times tried to solve the gyroscopic problems that were quite complex physical and mathematical tasks. The known gyroscope theories are met with skepticism due to simplifications and assumptions. Unsolved gyroscope problems presented a challenge for the researchers. The new analytical approaches to the gyroscopic inertial torques were published with mathematical errors that now are corrected. Today, the theory of gyroscopic effects was formulated by the fundamental principles of classical mechanics and opens a new direction for the dynamics of rotating objects. Science and engineering receive a new physical and analytical method for solving gyroscopic effects in engineering. The corrected mathematical models for the gyroscopic inertial torques can be useful in practice and present a good example for education processes.

Nomenclature

i:	Index for axis ox or oy
J:	Moment of inertia of a disc
J_i:	Moment of inertia of a disc around axis i
M:	Mass of a disc
m:	Mass element
R:	External radius of a disc
T:	Load torque
$T_{am \cdot i}$:	Torque generated by a change in an angular momentum acting around axis i
$T_{ct.i}$, $T_{cr \cdot i}$:	Torque generated by centrifugal and Coriolis forces, respectively, and acting around axis i
T_p:	Precession torque
T_r:	Resistance torque
$T_{t \cdot i}$:	Total torque acting around axis i
t:	Time
y_m,x_m:	Distance of the mass element's disposition relative to axes ox and oy
$y_{A,}$ y_C:	Centroid distance along with axis oy and ox, respectively
V:	Tangential velocity
α, δ:	Computational angles of a disc
γ:	Angle of a disc turn
ω:	Angular velocity of a disc
ω_i:	Angular velocity of a disc motion around axis i.

Data Availability

The research data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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