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Some Convergence Theorems of Henstock-Kurzweil-Dunford-Stieltjes Integral and Henstock-Kurzweil-Pettis-Stieltjes Integral of Banach-Valued Functions on $\mathbb{R}$

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Let X be an arbitrary Banach space. The establishment of the Henstock-Kurzweil-Dunford-Stieltjes (HKDS) Integral and Henstock-Kurzweil-Pettis-Stieltjes (HKPS) Integral of an X-valued function over $\mathbb{R}$ shows a viable and more generalized integration process utilizing the notion of dual spaces and weakly measurable

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functions. In this manuscript, the authors have discussed about some convergence theorems of Henstock-Kurzweil-Dunford-Stieltjes Integral and Henstock-Kurzweil-Pettis-Stieltjes Integral of X-valued functions on $\mathbb{R}$ via uniform convergence with respect to the integrand and integrator.

Keywords: HKDS integral; HKPS integral; bounded variation; uniform convergence; 2020.

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1 Introduction

For an arbitrary Banach space X with its corresponding dual and second dual space, X^* and X^{**}, it is known that an X-valued function f over a closed interval $[a, b]$ is said to be Henstock-Kurzweil-Dunford-Stieltjes integrable with respect to a function $g : [a, b] \to \mathbb{R}$ of bounded variation over $[a, b]$ if:

(i) For all $x^* \in X^*$, the function $x^* \circ f : [a, b] \to \mathbb{R}$ is HKS-integrable with respect to g on $[a, b]$.

(ii) For each compact subinterval $E \subset [a, b]$, there exists an element $x^{**}_E \in X^{**}$ such that

$$x^{**}_E \circ x^* = (\mathbf{HKS}) \int_E x^* \circ f \; dg$$

for all $x^* \in X^*$.

For a compact subinterval $E \subset [a, b]$, the value of HKDS-integral on E is

$$(\mathbf{HKDS}) \int_E f \; dg = x^{**}_E.[1]$$

On the other hand, if $f : [a, b] \to X$ is HKDS-integrable such that $(\mathbf{HKDS}) \int_E f \; dg \in X$, particularly $(\mathbf{HKDS}) \int_E f \; dg \in e(X)$, for every compact subinterval $E \subset [a, b]$, where e is the canonical embedding of X into X^{**}, then f is called Henstock-Kurzweil-Pettis-Stieltjes integrable with respect to g and

$$(\mathbf{HKPS}) \int_E f \; dg = (\mathbf{HKDS}) \int_E f \; dg$$

is called the HKPS-integral of f over the compact subinterval $E \subset [a, b]$ with respect to g. [1]

With these integrals, this article is devoted on constructing potential convergence theorems using the notion of uniform convergence supplementing our existing knowledge on HKDS-integral and HKPS-integral [2, 3, 4]. A sequence $\langle f_n \rangle_{n=1}^{\infty}$ of functions with common domain E, a function f on E and a subset A of E, we say that the sequence $\langle f_n \rangle_{n=1}^{\infty}$ converges to f uniformly on A provided that for each $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ for which

$$|f - f_n| < \varepsilon \text{ on } A$$

for all $n \geq \mathbb{N}$. [5]

2 Preliminary Notes

Essential terminologies needed in directing the conceptualization of the results are discussed on this section. Throughout the rest of the paper, we consider an arbitrary Banach space X [6]-[10].

Definition 2.1. [11] A **compact interval** in $\mathbb{R}$ is just a closed interval of the form $[u, v]$ where $u, v \in \mathbb{R}$. This interval is said to be **non-degenerate** if $u \neq v$.

Definition 2.2. [11] Two intervals $[u,v],[y,z] \in \mathbb{R}$ are said to be **non-overlapping** if

$$(u,v) \cap (y,z) = \varnothing.$$

Definition 2.3. [11] A function $\delta : [u,v] \to \mathbb{R}^+$ is called a **gauge** on $[u,v]$.

Definition 2.4. [11] A **point-interval pair** $(t,[u,v])$ consists of a point $t \in \mathbb{R}$ and an interval $[u,v]$ in $\mathbb{R}$. Here, t is known as a **tag** of $[u,v]$.

Definition 2.5. [11] If $\{([u_k,v_k]) : k = 1,2,\cdots,p\}$ is a finite collection of pairwise non-overlapping subintervals of $[a,b]$ such that $[a,b] = \bigcup_{k=1}^{p}[u_k,v_k]$, we say that $\{[u_k,v_k] : k = 1,2,\cdots,p\}$ is a **division** of $[a,b]$.

Definition 2.6. [11] A **Perron partition** P of $[a,b]$ is a finite collection of point-interval pairs $\{(t_k,[u_k,v_k]) : k = 1,2,\cdots,p\}$ where $\{[u_k,v_k\ :\ k = 1,2,\ldots,p]\}$ is a division of $[a,b]$ and $t_k \in [u_k,v_k]$ for $k = 1,2,\cdots,p$. Here, t_k is called a **tag** of $[u_k,v_k]$.

Definition 2.7. [11] Let δ be a gauge on $[a,b]$. A Perron partition $\{(t_k,[u_k,v_k]) : k = 1,2,\cdots,p\}$ of $[a,b]$ is **δ-fine** if $[u_k,v_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$.

Definition 2.8. [12] A function $g : [a,b] \to \mathbb{R}$ is said to be of **bounded variation** on $[a,b]$ if $\sup\left\{\sum_{[u_k,v_k]\in D} |g(u_k) - g(v_k)|\right\}$ is finite where the supremum is taken over all divisions $D = \{[u_k,v_k]\}$ of $[a,b]$.

Definition 2.9. [13] A normed space $(X,\|\cdot\|)$ is said to be **complete** if all Cauchy sequences in X are convergent. In this case, X is a **Banach space**.

Definition 2.10. [5] A sequence $\{f_n\}$ of real-valued functions on D is said to be **uniformly bounded on D** provided there is some $M > 0$ for which

$$|f_n| \leq M$$

on D for all $n \in \mathbb{N}$.

Definition 2.11. [13] An operator $T : V \to W$ between vector spaces V and W is a **linear operator** if for all $x,y \in V$ and scalars a,

$$T(x+y) = Tx + Ty \quad \text{and} \quad T(ax) = aTx.$$

Definition 2.12. [13] A **linear functional** is any linear operator $f : X \to K$, where X is a normed space over field K, where $K = \mathbb{R}$ or $K = \mathbb{C}$.

Definition 2.13. [13] Let X and Y be normed spaces and $T : \mathscr{D}(T) \to Y$ a linear operator, where $\mathscr{D}(T) \subset X$. The operator T is said to be **bounded** if there is a real number c such that for all $x \in \mathscr{D}(T)$,

$$\|Tx\| \leq c\|x\|.$$

Here, the smallest possible value c can take is observed on this inequality, $\frac{\|Tx\|}{\|x\|} \leq c$ where c must be at least as big as the supremum of

$$\left\{\frac{\|Tx\|}{\|x\|}\ :\ x \in \mathscr{D}(T)\right\}.$$

This quantity is denoted by $\|T\|$ and is called the **norm of the operator T**. If $c = \|T\|$, then

$$\|Tx\| \leq \|T\|\|x\|.$$

In case of linear functionals, we have

$$|f(x)| \leq \|f\|\|x\|.$$

Definition 2.14. [13] Let X be a vector space over K. Define

$X^* = \{f : X \to K \mid f \text{ is a linear functional}\}$ and the following operations in X^*,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x).$$

Then, $\langle X^*, +, \cdot \rangle$ is a vector space and is called the **algebraic dual space** of X.

Definition 2.15. [13] Let X be a vector space over field K. Define

$X^{**} = \{g : X^* \to K \mid g \circ f \text{ for all } f \in X^* \text{ where } g \text{ is a linear functional}\}$ and the following operations in X^{**},

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f \quad \text{and} \quad (\alpha g) \circ f = \alpha(g \circ f).$$

Then, $\langle X^{**}, +, \cdot \rangle$ is a vector space and is called the **second algebraic dual space** of X.

3 Main Results

The main results of this study is divided into two parts. The first part provides the convergence theorems of HKDS-integral and HKPS-integral of Banach-valued functions over $\mathbb{R}$ and the second part presents the Saks-Henstock lemma for these integrals [14]-[18].

3.1 Some Convergence Theorems

3.1.1 Uniform Convergence with respect to Integrand

Before presenting the uniform convergence for the integrand of HKDS-integral, we have the following lemma,

Lemma 3.1. *Let $f : [a, b] \to X$ be bounded linear operator and $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. If the **HKS**-integral of f with respect to g exists on $[a, b]$, then*

$$\left\| (\boldsymbol{HKS}) \int_E f \, dg \right\|_X \leq \|f\| \cdot M$$

for every compact subinterval $E \subset [a, b]$, where $M = \sup \left\{ \sum_{[u_k, v_k] \in D} |g(u_k) - g(v_k)| \right\}$.

Proof: Let $x^* \in X^*$. Since f is **HKS**-integrable with respect to g on $[a, b]$, then $(\mathbf{HKS}) \int_E f \, dg$ exists. Let $\varepsilon > 0$ and a compact subinterval $E \subset [a, b]$. Since f is **HKS**-integrable, choose a gauge δ on $[a, b]$ such that for every δ-fine Perron partition P of $[a, b]$, we have

$$\left\| S(f; g; P) - (\mathbf{HKS}) \int_E f \, dg \right\|_X < \varepsilon.$$

By hypothesis, $\|f\|$ exists. Let Q be a δ-fine Perron partition of $[a, b]$. Notice that,

$$\begin{aligned}
\big|\big|S(f;g;Q)\big|\big|_X &= \Bigg\| \sum_{(t_k,[u_k,v_k])\in Q} f(t_k)[(g(v_k)-g(u_k)]\Bigg\|_X \\
&\le \sum_{(t_k,[u_k,v_k])\in Q} \|f(t_k)[(g(v_k)-g(u_k)]\|_X \\
&= \sum_{(t_k,[u_k,v_k])\in Q} \|f(t_k)\|_X \cdot |g(v_k)-g(u_k)| \\
&\le \sum_{(t_k,[u_k,v_k])\in Q} \|f\| \cdot |g(v_k)-g(u_k)| \\
&= \|f\| \sum_{(t_k,[u_k,v_k])\in Q} |g(v_k)-g(u_k)| \\
&\le \|f\| \cdot M.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|(\mathbf{HKS})\int_E f\ dg\right\|_X &\le \left\|S\left(f;g;Q-(\mathbf{HKS})\int_E f\ dg\right)\right\|_X + \|S(f;g;Q)\|_X \\
&\le \varepsilon + \|f\| \cdot M.
\end{aligned}$$

By arbitrariness of ε, the conclusion follows. □

Theorem 3.2. ***(Uniform Convergence I)*** *Let $f_n : [a,b] \to X$ and let $g : [a,b] \to \mathbb{R}$ be a function of bounded variation. Suppose that $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of bounded and **HKDS**-integrable functions with respect to g over $[a,b]$. If f_n converges uniformly to $f : [a,b] \to X$ on $[a,b]$, then f is **HKDS**-integrable with respect to g over $[a,b]$ and*

$$\lim_{n\to\infty} (\boldsymbol{HKDS}) \int_E f_n\ dg = (\boldsymbol{HKDS}) \int_E f\ dg$$

for all compact subinterval $E \subset [a,b]$.

Proof: Let $\varepsilon > 0$ and $x^* \in X^*$. Note that f_n converges uniformly to f on $[a,b]$. So, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, and for all $h \in [a,b]$, we have

$$\|f_n(h) - f(h)\|_X < \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} \tag{1}$$

where $M = \sup\left\{\sum_{[u_k v_k]\in D} |g(u_k) - g(v_k)|\right\}$. If $m, n \ge N_1$ and $h \in [a,b]$, then

$$\begin{aligned}
\|f_n(h) - f_m(h)\|_X &= \|f_n(h) - f(h) + f(h) - f_m(h)\|_X \\
&\le \|f_n(h) - f(h)\|_X + \|f(h) - f_m(h)\|_X \\
&< \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} + \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)} \\
&= \frac{2 \cdot \varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)}.
\end{aligned}$$

Consequently, for all $m, n \ge N_1$, we have

$$\|f_n - f_m\| \le \frac{2 \cdot \varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M+1)}.$$

Now, from hypothesis, $\langle x^* \circ f_n \rangle_{n=1}^{\infty}$ is a sequence of **HKS**-integrable functions with respect to g over $[a, b]$ by Theorem 3.1.5 on [1]. Let $E \subset [a, b]$ be a compact subinterval. If $m, n \geq N_1$, then using Lemma 3.1 and by the linearity property of the integrand of **HKS**-integral, observe that

$$\left|(\mathbf{HKS}) \int_E x^* \circ f_n \, dg - (\mathbf{HKS}) \int_E x^* \circ f_m \, dg\right|$$
$$= \left|x^*\Big(\mathbf{HKS} \int_E f_n \, dg - \mathbf{HKS} \int_E f_m \, dg\Big)\right|$$

$$\begin{aligned}
&\leq \|x^*\|_{X^*} \cdot \left\|(\mathbf{HKS}) \int_E f_n \, dg - (\mathbf{HKS}) \int_E f_m \, dg\right\|_X \\
&= \|x^*\|_{X^*} \cdot \left\|(\mathbf{HKS}) \int_E (f_n - f_m) \, dg\right\|_X \\
&\leq \|x^*\|_{X^*} \cdot \|f_n - f_m\| \cdot M \\
&\leq \|x^*\|_{X^*} \frac{2 \cdot \varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M + 1)} \cdot M \\
&= \frac{2 \cdot \varepsilon}{3} < \varepsilon.
\end{aligned}$$

Hence, $\left\langle (\mathbf{HKS}) \int_E x^* \circ f_n \, dg \right\rangle_{n=1}^{\infty}$ is Cauchy. Since X is a Banach space, $\left\langle (\mathbf{HKS}) \int_E x^* \circ f_n \, dg \right\rangle_{n=1}^{\infty}$ converges to, say $A \in X$. Thus, there is an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\left\|(\mathbf{HKS}) \int_E x^* \circ f_n \, dg - A\right\|_X < \frac{\varepsilon}{3}.$$

Put $N = \max\{N_1, N_2\}$. Observe that $x^* \circ f_N$ is (**HKS**)-integrable with respect to g on $[a, b]$, so we can select a gauge δ such that for any δ-fine Perron partition P of $[a, b]$, we have

$$\left|S(x^* \circ f_N; g; P) - (\mathbf{HKS}) \int_E x^* \circ f_N \, dg\right| < \frac{\varepsilon}{3}.$$

Note that from (1), we have

$$\begin{aligned}
&\left|S(x^* \circ f; g; P) - S(x^* \circ f_N; g; P)\right| \\
&= \left|\sum_{(t_k, [u_k, v_k]) \in P} x^*(f(t_k))(g(v_k) - g(u_k)) - \sum_{(t_k, [u_k, v_k]) \in P} x^*(f_N(t_k))(g(v_k) - g(u_k))\right| \\
&= \sum_{(t_k, [u_k, v_k]) \in P} \left|x^*(f(t_k))[g(v_k) - g(u_k)] - x^*(f_N(t_k))[g(v_k) - g(u_k)]\right| \\
&\leq \sum_{(t_k, [u_k, v_k]) \in P} \left|x^*(f(t_k)) - x^*(f_N(t_k))\right| \cdot \left|g(v_k) - g(u_k)\right| \\
&\leq \sum_{(t_k, [u_k, v_k]) \in P} \|x^*\|_{X^*} \cdot \|f(t_k) - f_N(t_k)\|_X \cdot \left|g(v_k) - g(u_k)\right| \\
&\leq \|x^*\|_{X^*} \cdot \frac{\varepsilon}{3 \cdot (\|x^*\|_{X^*} + 1) \cdot (M + 1)} \cdot M \\
&= \frac{\varepsilon}{3}.
\end{aligned}$$

Hence,

$$\begin{aligned}\big|S(x^* \circ f; g; P) - A\big| &= \big|S(x^* \circ f; g; P) - S(x^* \circ f_N; g; P) + S(x^* \circ f_N; g : P) \\ &\quad - (\mathbf{HKS}) \int_E x^* \circ f_N \; dg + (\mathbf{HKS}) \int_E x^* \circ f_N \; dg - A\big| \\ &\le \big|S(x^* \circ f; g; P) - S(x^* \circ f_N; g; P)\big| + \Big|S(x^* \circ f_N; g; P) - (\mathbf{HKS}) \int_E x^* \circ f_N \; dg\Big| \\ &+ \Big|(\mathbf{HKS}) \int_E x^* \circ f_N \; dg - A\Big| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

This exhibits the **HKS**-integrability of $x^* \circ f$ with respect to g on $[a, b]$. So,

$$\lim_{n\to\infty} (\mathbf{HKS}) \int_E x^* \circ f_n \; dg = A = (\mathbf{HKS}) \int_E x^* \circ f \; dg.$$

By Theorem 3.1.5 on [1], f is **HKDS**-integrable with respect to g on $[a, b]$ for all $x^* \in X^*$. Now, for each $n \in \mathbb{N}$, put

$$x^{**}_{n,E}(x^*) = (\mathbf{HKS}) \int_E x^* \circ f_n \; dg.$$

Also,

$$x^{**}_E(x^*) = (\mathbf{HKS}) \int_E x^* \circ f \; dg.$$

This means that for all $x^* \in X^*$ and $n \in \mathbb{N}$, $x^{**}_{n,E}$ converges to x^{**}_E in X^{**}. Hence,

$$x^{**}_{n,E} = \lim_{n\to\infty} (\mathbf{HKDS}) \int_E f_n \; dg = (\mathbf{HKDS}) \int_E f \; dg = x^{**}_E.$$

□

For HKPS-integral, we have a similar convergence theorem,

Theorem 3.3. ***(Uniform Convergence I)*** *Let $f_n : [a, b] \to X$ and let $g : [a, b] \to \mathbb{R}$ be a function of bounded variation. Suppose that $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of bounded and **HKPS**-integrable functions with respect to g over $[a, b]$. If f_n converges uniformly to $f : [a, b] \to X$ on $[a, b]$, then f is **HKPS**-integrable with respect to g over $[a, b]$ and*

$$\lim_{n\to\infty} (\boldsymbol{HKPS}) \int_E f_n \; dg = (\boldsymbol{HKPS}) \int_E f \; dg$$

for all compact subinterval $E \subset [a, b]$.

Proof: Let $E \subset [a, b]$ be a compact subinterval. The assumption implies that f_n is **HKDS**-integrable with respect to g over $[a, b]$ such that $(\mathbf{HKDS}) \int_E f_n \; dg \in e(X)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, take $t_n \in X$ such that

$$e(t_n) = (\mathbf{HKDS}) \int_E f_n \; dg.$$

By Theorem 3.2, f is **HKDS**-integrable with respect to g over $[a, b]$ and

$$\lim_{n\to\infty} (\mathbf{HKDS}) \int_E f_n \; dg = (\mathbf{HKDS}) \int_E f \; dg$$

which implies $\lim_{n\to\infty} e(t_n) = (\mathbf{HKDS}) \int_E f\ dg \in e(X)$. This indicates that f is **HKPS**-integrable with respect to g on $[a.b]$. Consequently, put $t \in X$ such that

$$e(t) = (\mathbf{HKDS}) \int_E f\ dg$$

and the equality follows. □

3.1.2 Uniform Convergence with respect to Integrator

Let's proceed to the uniform convergence with respect to the integrator and it needs the following lemmas,

Theorem 3.4. *Let $g, g_n : [a, b] \to \mathbb{R}$ and $\langle g_n\rangle_{n=1}^{\infty}$ be a sequence of functions such that g_n converges uniformly to g and g_n is uniformly bounded. If g_n is a bounded variation on $[a, b]$ for all $n \in \mathbb{N}$, then g is also a bounded variation on $[a, b]$.*

Proof: Let $g : [a, b] \to \mathbb{R}$ and $\langle g_n\rangle_{n=1}^{\infty}$ be a sequence of functions on $[a, b]$. By assumption, for each $n \in \mathbb{N}$, g_n is a bounded variation on $[a, b]$. This means that for each $n \in \mathbb{N}$,

$$\sup_n \left\{ \sum_{[u_k,v_k]\in D} |g_n(u_k) - g_n(v_k)| \right\} < \infty.$$

This implies that $\sum_{[u_k,v_k]\in D} |g_n(u_k) - g_n(v_k)| < \infty$ for each $n \in \mathbb{N}$. Let S be a division of $[a, b]$ and let $M > 0$ such that $|g_n| \leq M$. Note that

$$\sum_{[u,v]\in S} |g(u) - g(v)| = \sum_{[u_k,v_k]\in S} \left| \lim_{n\to\infty} g_n(u_k) - \lim_{n\to\infty} g_n(v_k) \right|$$

since g_n converges uniformly to g on $[a, b]$. Now,

$$\begin{aligned}
\sum_{[u,v]\in S} |g(u) - g(v)| &= \sum_{[u_k,v_k]\in S} \left| \lim_{n\to\infty} g_n(u_k) - \lim_{n\to\infty} g_n(v_k) \right| \\
&= \sum_{[u_k,v_k]\in S} \left(\lim_{n\to\infty} |g_n(u) - g_n(v)| \right) \\
&= \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(u_k) - g_n(v_k)| \right) \\
&\leq \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(u_k)| + |g_n(v_k)| \right) \\
&= \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(u_k)| \right) + \lim_{n\to\infty} \left(\sum_{[u_k,v_k]\in S} |g_n(v_k)| \right) \\
&\leq \lim_{n\to\infty} \sum_{[u_k,v_k]\in S} M + \lim_{n\to\infty} \sum_{[u_k,v_k]\in S} M \\
&= \lim_{n\to\infty} \sum_{[u_k,v_k]\in S} 2M = k \cdot 2M < \infty
\end{aligned}$$

where k is the number of subintervals on S. Fix $M_o = k \cdot 2M$. Then $\sum_{[u,v]\in S} |g(u) - g(v)| \leq M_o$. Taking the supremum, we have

$$\sup \left\{ \sum_{[u,v]\in S} |g(u) - g(v)| \right\} \leq M_o < \infty.$$

Therefore, g is a bounded variation on $[a, b]$. □

Theorem 3.5. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence of functions that are of bounded variation. If g_n converges uniformly to g and* $\sup\{|D| : D$ *is a division of* $[a, b]\}$ *is finite. Then the sequence*

$$\left\langle (\boldsymbol{HKS}) \int_{[a,b]} x^*(f) \; dg_n \right\rangle_{n=1}^{\infty}$$

is Cauchy for all $x^* \in X^*$.

Proof: Let $x^* \in X^*$. Note that Proposition 3.3.2 on [1] states that $x^* \circ f$ is continuous on $[a, b]$. Using Lemma 3.3.1 on [1], $x^* \circ f$ is **HKS**-integrable with respect to g_n on $[a, b]$ for all $x^* \in X^*$. Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, there exists a gauge δ_n such that

$$\left| S(x^* \circ f; g_n; P_n) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg_n \right| < \frac{\varepsilon}{4}$$

for every δ_n-fine Perron partition P_n of $[a, b]$. Put $\delta = \inf\{\delta_n \; : \; n \in \mathbb{N}\}$. Let P be a δ-fine Perron partition of $[a, b]$. Then P is a δ_n-fine Perron partition of $[a, b]$ for all $n \in \mathbb{N}$. This implies

$$\left| S(x^* \circ f; g_n; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg_n \right| < \frac{\varepsilon}{4}.$$

Since $x^* \circ f$ is continuous on $[a, b]$, $x^* \circ f$ is bounded in $[a, b]$. This implies an existence of $K > 0$ such that $|x^*(f(h))| \leq K$ for all $h \in [a, b]$. Now, put

$$W = \sup\{|D| \; : \; D \text{ is a division of } [a, b]\}.$$

Since g_n converges uniformly to g on $[a, b]$, there is $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$ and $h \in [a, b]$, we have

$$|g_n(h) - g(h)| \leq \frac{\varepsilon}{16(K + 1)(W + 1)}.$$

By Lemma 3.4, g is a function of bounded variation on $[a, b]$. Also, $g_n - g$ is a function of bounded variation on $[a, b]$. Let D be a division of $[a, b]$. We will now find a bound for

$$\sum_{[u_k,v_k]\in D} |(g_n-g)(u_k)-(g_n-g)(v_k)|,$$

$$\begin{aligned}
&\sum_{[u_k,v_k]\in D} |(g_n-g)(u_k)-(g_n-g)(v_k)| \\
&\quad \le \sum_{[u_k,v_k]\in D} |(g_n-g)(u_k)| + \sum_{[u_k,v_k]\in D} |(g_n-g)(v_k)| \\
&\quad = \sum_{[u_k,v_k]\in D} |g_n(u_k)-g(u_k)| + \sum_{[u_k,v_k]\in D} |g_n(v_k)-g(v_k)| \\
&\quad \le \sum_{[u_k,v_k]\in D} \frac{\varepsilon}{8(K+1)(W+1)} \\
&\quad = |D|\cdot \frac{\varepsilon}{8(K+1)(W+1)} \\
&\quad \le W\cdot \frac{\varepsilon}{8(K+1)(W+1)} = \frac{\varepsilon}{8(K+1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|S(x^*\circ f; g_n; P) - S(x^*\circ f; g; P)| \\
&\quad = |S(x^*\circ f; g_n-g; P)| \\
&\quad = \left| \sum_{(t_k,[u_k,v_k])\in P} x^*(f(t_k))[(g_n-g)(v_k)-(g_n-g)(u_k)] \right| \\
&\quad \le \sum_{(t_k,[u_k,v_k])\in P} |x^*(f(t_k))[(g_n-g)(v_k)-(g_n-g)(u_k)]| \\
&\quad \le \sum_{(t_k,[u_k,v_k])\in P} |x^*(f(t_k))|\cdot|[(g_n-g)(v_k)-(g_n-g)(u_k)]| \\
&\quad \le \sum_{(t_k,[u_k,v_k])\in P} K\cdot|(g_n-g)(v_k)-(g_n-g)(u_k)| \\
&\quad = K\cdot \sum_{(t_k,[u_k,v_k])\in P} |(g_n-g)(v_k)-(g_n-g)(u_k)| \\
&\quad \le K\cdot \frac{\varepsilon}{8(K+1)} = \frac{\varepsilon}{8}.
\end{aligned}$$

So, if $m, n \ge N$, then

$$\begin{aligned}
&|S(x^*\circ f; g_n; P) - S(x^*\circ f; g_m; P)| \\
&\quad \le |S(x^*\circ f; g_n; P) - S(x^*\circ f; g; P)| + |S(x^*\circ f; g; P) - S(x^*\circ f; g_m; P)| \\
&\quad = |S(x^*\circ f; g_n-g; P)| + |S(x^*\circ f; g_m-g; P)| \\
&\quad < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \\
&\quad = \frac{\varepsilon}{4}.
\end{aligned}$$

Therefore, for all $m, n \geq N$,

$$\begin{aligned}
&\left|(\mathbf{HKS})\int_{[a,b]} x^* \circ f \; dg_n - (\mathbf{HKS})\int_{[a,b]} x^* \circ f \; dg_m\right| \\
&= \left|(\mathbf{HKS})\int_{[a,b]} x^* \circ f \; dg_n - S(x^* \circ f; g_n; P)\right. \\
&\quad + S(x^* \circ f; g_n; P) - S(x^* \circ f; g; P) \\
&\quad\quad + S(x^* \circ f; g; P - S(x^* \circ f; g_m; P) \\
&\quad\quad\quad \left. + S(x^* \circ f; g_m; P) - (\mathbf{HKS})\int_{[a,b]} x^* \circ f \; dg_m\right| \\
&\leq \left|(\mathbf{HKS})\int_{[a,b]} x^* \circ f \; dg_n - S(x^* \circ f; g_n; P\right| \\
&\quad + |S(x^* \circ f; g_n; P) - S(x^* \circ f; g; P)| \\
&\quad\quad + |S(x^* \circ f; g; P - S(x^* \circ f; g_m; P)| \\
&\quad\quad\quad + \left|S(x^* \circ f; g_m; P) - (\mathbf{HKS})\int_{[a,b]} x^* \circ f \; dg_m\right| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \varepsilon
\end{aligned}$$

which implies that the sequence $\left\langle (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n \right\rangle_{n=1}^{\infty}$ is Cauchy. □

Theorem 3.6. ***(Uniform Convergence II)*** *Let $f : [a, b] \to X$ be a continuous function on $[a, b]$ and let $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence of functions on $[a, b]$ that are bounded variation. Suppose that g_n converges uniformly to g on $[a, b]$, then f is **HKDS**-integrable with respect to g on $[a, b]$ and*

$$\lim_{n\to\infty} (\boldsymbol{HKDS}) \int_E f \; dg_n = (\boldsymbol{HKDS}) \int_E f \; dg$$

for all compact subinterval $E \subset [a, b]$.

Proof: Let $x^* \in X^*$. Using Lemma 3.5, the sequence $\left\langle (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n \right\rangle_{n=1}^{\infty}$ is Cauchy. Consequently, this sequence converges, so we can fix $\lim_{n\to\infty} (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n = K$. It remains to show that $K = (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg$. Being convergent implies the existence of $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|(\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_n - A\right| < \frac{\varepsilon}{3}.$$

Specifically,

$$\left|(\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_N - A\right| < \frac{\varepsilon}{3}. \tag{2}$$

Since $x^* \circ f$ is **HKS**-integrable with respect to g_N on $[a, b]$, we can choose a gauge δ on $[a, b]$ such that

$$\left|S(f; g_N; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg_N\right| \tag{3}$$

for any δ-fine Perron partition P on $[a,b]$. Furthermore, using the part of the proof of Lemma 3.5, we can have,

$$|S(x^* \circ f; g; P) - S(x^* \circ f; g_N; P)| < \frac{\varepsilon}{3}. \tag{4}$$

Hence, by (1),(2), and (3), we have,

$$\begin{aligned}
|S(x^* \circ f; g; P) - K| &= \Bigg| S(x^* \circ f; g; P) - S(x^* \circ f; g_N; P) \\
&\quad + S(x^* \circ f; g_N; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg \\
&\quad + (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg - K \Bigg| \\
&\leq |S(x^* \circ f; g; P) - S(x^* \circ f; g_N; P)| \\
&\quad + \left| S(x^* \circ f; g_N; P) - (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg \right| \\
&\quad + \left| (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg - K \right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Thus, $K = (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \;\; dg$ indicating that $x^* \circ f$ is **HKS**-integrable with respect to g on $[a,b]$. So,

$$\lim_{n\to\infty} (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg_n = (\mathbf{HKS}) \int_{[a,b]} x^* \circ f \; dg.$$

To this end, by Theorem 3.1.5 on [1], f is **HKDS**-integrable with respect to g and g_n on $[a,b]$. Now, let a compact subinterval $E \subset [a,b]$. For each $n \in \mathbb{N}$, put

$$x^{**}_{n,E}(x^*) = (\mathbf{HKS}) \int_E x^* \circ f \; dg_n.$$

Also,

$$x^{**}_E(x^*) = (\mathbf{HKS}) \int_E x^* \circ f \; dg.$$

This means that for all $x^* \in X^*$ and $n \in \mathbb{N}$, $x^{**}_{n,E}$ converges to x^{**}_E in X^{**}. Finally,

$$x^{**}_{n,E} = \lim_{n\to\infty} (\mathbf{HKDS}) \int_E f \; dg_n = (\mathbf{HKDS}) \int_E f \; dg = x^{**}_E.$$

□

On Pettis type integral, we have the following uniform convergence with respect to the integrator.

Theorem 3.7. *Let $f : [a,b] \to X$ be a continuous function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ such that $(\boldsymbol{HKDS}) \int_E f \; dg \in e(X)$ and let $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence of functions on $[a,b]$ that are of bounded variation such that $(\boldsymbol{HKDS}) \int_E f \; dg_n \in e(X)$. Suppose that g_n converges uniformly to g on $[a,b]$, then*

$$\lim_{n\to\infty} (\boldsymbol{HKPS}) \int_E f \; dg_n = (\boldsymbol{HKPS}) \int_E f \; dg$$

for all compact subinterval $E \subset [a,b]$.

Proof: Let E be a compact subinterval of $[a, b]$. By hypothesis, $(\mathbf{HKDS}) \int_E f \; dg \in e(X)$ implies f being **HKPS**-integrable with respect to g on $[a, b]$. In a similar manner, for each $n \in \mathbb{N}$, $(\mathbf{HKDS}) \int_E f \; dg_n \in e(X)$ implying that f is **HKPS**-integrable with respect to g_n on $[a, b]$. Fix $u, u_n \in X$ such that

$$e(u) = (\mathbf{HKDS}) \int_E f \; dg \quad \text{and} \quad e(u_n) = (\mathbf{HKDS}) \int_E f \; dg_n.$$

Observe that by Theorem 3.6,

$$\lim_{n \to \infty} (\mathbf{HKDS}) \int_E f \; dg_n = (\mathbf{HKDS}) \int_E f \; dg.$$

This indicates that

$$\lim_{n \to \infty} e(u_n) = e(u).$$

That is, the sequence $\langle e(u_n) \rangle_{n=1}^{\infty}$ in $e(X)$ converges to $e(u)$. Consequently, the claimed equality follows by definition of **HKPS** integral. □

4 Conclusion

Let X be a Banach space. Given a sequence of Banach-valued functions $\langle f_n \rangle_{n=1}^{\infty}$ on $\mathbb{R}$, the presentation of convergence theorems for HKDS integral and HKPS integral using the notion of uniform convergence with respect to the integrand and integrator provide sufficient conditions for a Banach-valued function f on $\mathbb{R}$ to be integrable with respect to this sequence [19]-[21]. This is vital especially on predicting the integral values of such functions efficiently and systematically.

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Competing Interests

Authors have declared that no competing interests exist.

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