Asian Journal of Mathematics and Computer Research

Volume 31, Issue 4, Page 1-9, 2024; Article no.AJOMCOR.12105
ISSN: 2456-477X

About Supports of Local Cohomology Modules

Carlos Henrique Tognon [a*]

[a]*Department of Mathematics, University of São Paulo, ICMC, São Carlos, SP, Brazil.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: https://doi.org/10.56557/ajomcor/2024/v31i48864

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://prh.ikprress.org/review-history/12105

Received: 15/03/2024
Accepted: 22/05/2024
Published: 21/09/2024

Original Research Article

Abstract

The article provides the relation between the theory of local cohomology modules, and vanishing results, and also about the theory of support of such modules. Here, we put results about the theory, and also we provide a relation of local cohomology in the theory of commutative algebra and homological algebra.

Keywords: Local cohomology modules; support; noetherian ring; depth.

2010 Mathematics Subject Classification: 13C13, 13D45.

1 Introduction

Here in this paper, R is a commutative Noetherian ring with non-zero identity.

Thus, for R be a Noetherian ring, let I be an ideal of R, and let $I(G)$ be an R-module, which is the edge ideal, that we will see in the prerequisites.

**Corresponding author: E-mail: carlostognon@gmail.com;*

Cite as: *Tognon, Carlos Henrique. 2024. "About Supports of Local Cohomology Modules". Asian Journal of Mathematics and Computer Research 31 (4):1-9. https://doi.org/10.56557/ajomcor/2024/v31i48864.*

The local cohomology of $I(G)$ with respect to I, which can be called of local cohomology of the edge ideal, was introduced by Grothendieck by

$$\mathrm{H}_I^n(I(G)) = \varinjlim_{l\in\mathbb{N}} \mathrm{Ext}_R^n(R/I^l, I(G)).$$

See about local cohomology modules, in [1].

Here, we put results for the local cohomology module defined by an ideal.

Let

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I \subseteq \mathfrak{p}\}.$$

The set of elements x of $I(G)$ such that $\mathrm{Supp}_R(Rx) \subseteq V(I)$ is said to be I-torsion submodule of $I(G)$ and is denoted by $\Gamma_I(I(G))$.

Note that $\Gamma_I(\bullet)$ is a covariant, R-linear functor from the category of R-modules to itself.

For an integer i, the local cohomology functor $\mathrm{H}_I^i(\bullet)$ with respect to I is defined to be the i-th right derived functor of $\Gamma_I(\bullet)$.

Also $\mathrm{H}_I^i(I(G))$, according to [1], is called the i-th local cohomology module of edge ideal $I(G)$, with respect to I.

By [2, Remark 3.5.3 (a)], we have $\mathrm{H}_I^0\left(I(G)\right) \cong \Gamma_I\left(I(G)\right)$, where we have that

$$\Gamma_I\left(I(G)\right) := \left\{m \in I(G) \mid I^t m = 0 \text{ for some } t \in \mathbb{N}\right\},$$

is an R-submodule of the R-module $I(G)$.

We can also to see this definition of the following form.

([2, Definition 3.5.2]) The local cohomology functors, denoted by $\mathrm{H}_I^i\left(\bullet\right)$, are the right derived functors of $\Gamma_I\left(\bullet\right)$. In other words, if $\mathbf{I}^\bullet$ is an injective resolution of the R-module M, then $\mathrm{H}_I^i\left(M\right) \cong \mathrm{H}^i\left(\Gamma_I\left(\mathbf{I}_M^\bullet\right)\right)$ for all $i \geq 0$, where $\mathbf{I}_M^\bullet$ denotes the deleted injective resolution of M.

Local cohomology theory, as in the Definition 1, has been a significant tool in commutative algebra and algebraic geometry.

In the Section 2, we put about graphs theory.
In the Section 3, we put some prerequisites.
In the Section 4, we presented some results about the theory in question.
In the Section 5, we presented applications.
We finalize the paper with a conclusion.

The reader should consult either [1], or [2], for any unexplained notation or terminology.

For the development of the results, see [3] and [4].

2 Graphs Theory

Let us present in this section the concepts of the graphs theory that we will use in the course of this work.

Here, is in accordance with [5] and [6].

Let $R = K[v_1, \ldots, v_s]$ be a polynomial ring over a field K, and let $Z = \{z_1, \ldots, z_q\}$ be a finite set of monomials in R.

The **monomial subring** spanned by Z is the K-subalgebra,

$$K[Z] = K[z_1, \ldots, z_q] \subset R.$$

Thus, consider any graph G, simple and finite without isolated vertices, with vertex set $V(G) = \{v_1, \ldots, v_s\}$.

Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $i \neq j$, in $R = K[v_1, \ldots, v_s]$, such that $\{v_iv_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the edges of the graph G.

If G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G, over some fixed field K.

([5]) According to the previous context, the **edge ideal** of a finite simple graph G, with no isolated vertices, is defined by

$$I(G) = (v_iv_j \mid v_iv_j \text{ is an edge of } G),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

We take K a fixed field, and we consider $K[v_1, v_2 \ldots, v_s]$ the ring polynomial over the field K.

Since K is a field, we have that K is a Noetherian ring. Thus, we have then that $K[v_1, \ldots, v_s]$ is also a Noetherian ring (by Theorem of the Hilbert Basis).

Remark 2.1. By the previous context, $R = K[v_1, v_2 \ldots, v_s]$ is a Noetherian ring. Thus, the edge ideal $I(G)$ is an R-module, and thus we can to get characterizations for this module $I(G)$ under certain hypothesis.

And let's denote $I(G)$ by $G^{'}$.

3 Prerequisites

Throughout this paper $I(G)$, as previously stated, is a finitely generated module over a Noetherian ring R. Let $\mathrm{pd}_R(R)$ denote the projective dimension of R.

For the ideal $I^{'} = (v_1, \ldots, v_s)$ of R, we denote by $I^{'}_R = \mathrm{ann}_R(R/I^{'})$ the annihilator of the module $R/I^{'}$ and by $\Gamma_{I^{'}}(\bullet)$ the $I^{'}$-torsion functor.

Lemma 3.1. ([7, Lemmas 2.1, 2.3]) *The following statements are true.*

(1) *Let $E^{\bullet}$ be an injective resolution of $G^{'}$. Then, for any $j \geq 0$, we have*

$$\mathrm{H}^j_{I^{'}}(G^{'}) \cong \mathrm{H}^j(\Gamma_{I^{'}}(E^{\bullet})) \cong \mathrm{H}^j(\Gamma_{I^{'}}(E^{\bullet})) \cong \mathrm{H}^j(\Gamma_{I^{'}_R}(E^{\bullet})).$$

(2) *If* $\Gamma_{I'_R}(G') = G'$ *or* $I' \subseteq \mathrm{ann}_R(R)$, *then* $\mathrm{H}^j_{I'}(G') \cong \mathrm{Ext}^j_R(R, G')$, *for all* $j \geq 0$.

Lemma 3.2. ([7, Theorem 2.4]) *Let* $l = \mathrm{depth}(I'_R, G')$. *Then*

$$\mathrm{Ass}_R(\mathrm{H}^l_{I'}(G') = \mathrm{Ass}_R(\mathrm{Ext}^l_R(R/I', G').$$

Lemma 3.3. ([8, Theorem 3.7]) *If* $\mathrm{pd}_R(R) < \infty$, *then* $\mathrm{H}^j_{I'}(G') = 0$, *for all* $j > \mathrm{pd}_R(R) + \dim(R \otimes_R G')$.

Lemma 3.4. ([9, Lemma 3.1]) *Let* $d = \dim(R)$. *If* $\mathrm{pd}_R(R) < \infty$, *then*

$$\dim(\mathrm{Ext}^j_R(R, R)) \leq d - j,$$

for all $0 \leq j \leq \mathrm{pd}_R(R)$.

4 The Main Results

Here, we have some results.

Suppose that the local ring homomorphism $f : R \to S$ is flat. Then, we have that

$$\mathrm{H}^j_I(I(G)) \otimes_R S \cong \mathrm{H}^j_{IS}(R \otimes_R S, I(G) \otimes_R S),$$

for all $j \geq 0$.

Proof. See [1], as reference. □

Let $n = \dim(G')$. Then

$$\mathrm{Supp}_R(\mathrm{H}^{n-1}_I(G')$$

is a set, which is finite.

Proof. Let $J = \mathrm{ann}_R(G')$ and $R' = R/J$, and then

$$\dim(R') = n,$$

and G' is an R'-module. Hence, by the independence theorem in [1], we have

$$\mathrm{H}^{n-1}_{I'}(G') \cong \mathrm{H}^{n-1}_{I'R'}(G')$$

as R-modules.

Thus, by [10, Corollary 2.5],

$$\mathrm{Supp}_{R'}(\mathrm{H}^{n-1}_{I'R'}(G')$$

is a set, which is finite.

Also, we have that

$$\mathrm{Supp}_R(\mathrm{H}^{n-1}_{I'}(G^{'}) \subseteq \mathrm{Supp}_R(R/J),$$

and

$$\mathrm{Supp}_{\bar{R}}(\mathrm{H}^{n-1}_{I\bar{R}}(I(G))) = \left\{\mathfrak{p}/J \mid \mathfrak{p} \in \mathrm{Supp}_R(\mathrm{H}^{n-1}_I(I(G)))\right\}.$$

Therefore,

$$\mathrm{Supp}_R(\mathrm{H}^{n-1}_I(I(G)))$$

is a set, which is finite. □

We have now the following result.

Let $i \in \mathbb{N} \cup \{+\infty\}$.

Set

$$J_i = \bigcap_{j<i} \mathrm{ann}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}).$$

Then,

$$\mathrm{H}^j_{I'}(G^{'}) \cong \mathrm{H}^j_{J_i}(G^{'}),$$

for any $j < i$.

Proof. Note that $I^{'}_R \subseteq J_i$. Let

$$E^{\bullet} : 0 \to E^0 \to \ldots \to E^j \to \ldots$$

be a minimal injective resolution of $G^{'}$.

So, for $j \geq 0$, by [1, 10.1.10],

$$\Gamma_{I^{'}_R}(E^j) = \bigoplus_{I^{'}_R \subseteq I \in \mathrm{Ass}_R(E^j)} E(R/I)^{\mu^j}(I, G^{'}) \quad ,$$

and,

$$\Gamma_{J_i}(E^j) = \bigoplus_{J_i \subseteq I \in \mathrm{Ass}_R(E^j)} E(R/I)^{\mu^j(I,G^{'})}$$

where $\mu^j(I, G^{'}) = \dim_{k(I)}(\mathrm{Ext}^j_{R_I}(k(I), G^{'}_I))$ is j-th Bass number of $G^{'}$ with respect to I. Note that for any $I \in \mathrm{Ass}_R(E^j)$ the sequence

$$0 \to E^0_I \to E^1_I \to \ldots \to E^j_I \ldots$$

is a minimal injective resolution of $G^{'}_I$ (cf. [1, 11.1.6]).

So, as $E^j_I \neq 0$, $G^{'}_I \neq 0$.

Now, we have for $i \in \mathbb{N}$, and, $i = +\infty$.

Let $i \in \mathbb{N}$. For all $j < i$, and any $I \in \mathrm{Ass}_R(E^j)$ such that $I^{'}_R \subseteq I$ and J_i not is contained in I, we have that

$$\mathrm{Ext}^l_R(R/I^{'}, G^{'})_I = 0,$$

for all $l < i$.

It implies $\mathrm{depth}((I^{'}_R)_I, G^{'}_I) \geq i$, so $\mathrm{depth}(G^{'}_I) \geq i$.

So, $\mu^j(I, G^{'}) = 0$, so that $\Gamma_{I^{'}_R}(E^j) = \Gamma_{J_i}(E^j)$. Hence, by Lemma 3.1, we get

$$\mathrm{H}^j_{I^{'}}(G^{'}) \cong \mathrm{H}^j_{J_i}(G^{'}),$$

for any $j < i$.

Finally, if $i = +\infty$, then

$$J_i = \bigcap_{j \geq 0} \mathrm{ann}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}).$$

For all $j \geq 0$, and any $I \in \mathrm{Ass}_R(E^j)$ such that $I^{'}_R \subseteq I$, we obtain that $(I^{'}_R)_I G^{'}_I \neq G^{'}_I$. Set $\varpi = \mathrm{depth}((I^{'}_R)_I, G^{'}_I)$, then $\varpi < +\infty$, and

$$I \in \mathrm{Supp}_R(\mathrm{Ext}^{\varpi}_R(R/I^{'}, G^{'}).$$

It follows that,

$$J_i \subseteq \mathrm{ann}_R(\mathrm{Ext}^{\varpi}_R(R/I^{'}, G^{'}) \subseteq I.$$

And hence, we have that $\Gamma_{I^{'}_R}(E^j) = \Gamma_{J_i}(E^j)$.

Therefore, by Lemma 3.1, we obtain that

$$\mathrm{H}^j_{I^{'}}(G^{'}) \cong \mathrm{H}^j_{J_i}(G^{'}),$$

for any $j \geq 0$. □

5 Applications

Theorem 5.1. *For $i \in \mathbb{N} \cup \{+\infty\}$, we have*

$$\bigcup_{j<i} \mathrm{Supp}_R(\mathrm{H}^j_{I^{'}}(G^{'}) = \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}).$$

Proof. For $i \in \mathbb{N} \cup \{+\infty\}$, we take

$$J_i = \bigcap_{j<i} \mathrm{ann}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}). \qquad (1)$$

Then, by Proposition 4, we obtain that

$$\mathrm{H}^j_{I^{'}}(G^{'}) \cong \mathrm{H}^j_{J_i}(G^{'})$$

for any $j < i$.

So, we have that

$$\bigcup_{j<i} \mathrm{Supp}_R(\mathrm{H}^j_{I'}(G^{'}) \subseteq \mathrm{Supp}_R(R/J_i) = \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}). \qquad (2)$$

Also, let

$$I \in \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}). \qquad (3)$$

Set $\varpi = \mathrm{depth}((I^{'}_R)_I, G^{'}_I)$, then $\varpi < i$. For all $n > 0$, we have

$$I^{'}_R \subseteq \sqrt{\mathrm{ann}_R(I'/I'^n)},$$

so that

$$\mathrm{Ext}^j_R(I^{'}/I^{'^n}, G^{'})_I = 0,$$

for any $j < \varpi$.
Thus, the sequence

$$0 \to \mathrm{Ext}^{\varpi}_R(R/I^{'}, G^{'})_I \to \mathrm{Ext}^{\varpi}_R(R/I^{i^n}, G^{'})_I \qquad (4)$$

is exact for any $n > 0$.

So, we have an exact sequence

$$0 \to \mathrm{Ext}^{\varpi}_R(R/I^{'}, G^{'})_I \to \mathrm{H}^{\varpi}_{I'}(G^{'})_I\ .$$

As $\mathrm{Ext}^{\varpi}_R(R/I^{'}, G^{'})_I \neq 0$ and $\varpi < i$, we have that

$$I \in \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{H}^j_{I'}(G^{'}).$$

□

We have the following corollary.

Corollary 5.2. *Let* $n = \dim(G^{'})$. *Then*

$$\bigcup_{j\geq 0} \mathrm{Supp}_R(\mathrm{H}^j_{I'}(G^{'}) = \bigcup_{j\leq n} \mathrm{Supp}_R(\mathrm{H}^j_{I^{'}_R}(G^{'}) = \bigcup_{j\leq n} \mathrm{Supp}_R(\mathrm{H}^j_{I'}(G^{'}).$$

Proof. For $i \in \mathbb{N} \cup \{+\infty\}$, by Theorem 5.1,

$$\bigcup_{j<i} \mathrm{Supp}_R(\mathrm{H}^j_{I^{'}_R}(G^{'}) = \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/I^{'}_R, G^{'}). \qquad (5)$$

Moreover, we get

$$\bigcup_{j<i} \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/I^{'}_R, G^{'}) = \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/I^{'}, G^{'}). \qquad (6)$$

So, by Theorem 5.1,

$$\bigcup_{j<i} \mathrm{Supp}_R(\mathrm{H}^j_{I'_R}(G')) = \bigcup_{j<i} \mathrm{Supp}_R(\mathrm{H}^j_{I'}(G')), \qquad (7)$$

and so the result it follows by Grothendiecks Vanishing Theorem. $\square$

6 Conclusion

We relate here, the theory of commutative algebra and homological algebra, to the theory of local cohomology.

Moreover, in the course of the results we associate the support of local cohomology module, with the extension functor of this module.

We put applications about the support of this module.

Note that the theory in question, has also been studied by the following authors [11], [12], [13], [14], [15].

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Acknowledgement

I thank the referees for the comments that helped me improve my article.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Brodmann MP, Sharp RY. Local cohomology: An algebraic introduction with geometric applications. Cambridge studies in Advanced Mathematics, 60, Cambridge University Press, Cambridge; 1998.

[2] Bruns W, Herzog J. Cohen-Macaulay rings. University of Oxford, Oxford, London UK: 1997;1-150.

[3] Atiyah MF, Macdonald IG. Introduction to Commutative Algebra. University of Oxford; 1969.

[4] Rotman JJ. An introduction to homological algebra. University of Illinois, Urbana, Academic Press; 1979.

[5] Alilooee A, Banerjee A. Powers of edge ideals of regularity three bipartite graphs. Journal of Commutative Algebra. 2017;9:441-454.

[6] Simis A, Vasconcelos WV, Villarreal RH. The integral closure of subrings associated to graphs. Journal of Algebra. 1998;199:281-289.

[7] Cuong NT, Hoang NV. Some finite properties of generalized local cohomology modules, East-West J. Math. 2005;7:107-115.

[8] Yassemi S. Generalized section functors. J. Pure Appl. Alg. 1994;95:103-119.

[9] Herzog J, Zamani N. Duality and vanishing of generalized local cohomology. Arch. Math. J. 2003;81:512-519.

[10] Marley Th. Associated primes of local cohomology module over rings of small dimension. Manuscripta Math. 2001;104:519-525.

[11] Asadollahi J, Khashyarmanesh K, Salarian Sh. On the finiteness properties of the generalized local cohomology modules. Comm. Algebra. 2002;30:859-867.

[12] Divaani-Aazar K, Mafi A. Associated primes of local cohomology modules. Proc. Amer. Math. Soc. 2005;133:655 - 660.

[13] Nhan LT. On generalized regular sequences and the finiteness for associated primes of local cohomology modules, Comm. Algebra. 2005;33:793-806.

[14] Tognon CH. Local cohomology and maximal generalized modules. Asian Journal of Mathematics and Computer Research. 2024;31:1-7.

[15] Tognon CH. On the noetherian dimension of local cohomology modules. Far East Journal of Mathematical Sciences: FJMS. 2023;140:257-274.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://prh.ikprress.org/review-history/12105