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Research Article

Analysis of Fractional Thin Film Flow of Third Grade Fluid in Lifting and Drainage via Homotopy Perturbation Procedure

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Analysis of thin film flows is an important topic in fluid dynamics due to the large number of industrial applications such as food processing, chip manufacturing, irrigation, oil refining process, painting finishing, etc. Analysis involves studying the effects of various parameters in absolute conditions. These parameters may be film thickness, volumetric flux, liquid velocity profile, viscosity, shear stress, gravity, density, and different boundary formations. We have expanded the formulations of non-Newtonian third grade fluid for lifting and draining in fractional space. Fractional calculus along with Homotopy Perturbation Method is used for the solution and analysis purposes. The suitability and consistency of the solutions is determined by detecting residuals in each case. Velocity profile, average velocity, and volume flow for lifting and drainage cases are calculated. To the best of authors knowledge, thin film flow of fractional third grade fluid is not attempted before in lifting and drainage. Investigation shows increase in value of fractional parameter that decreases the velocity profile in lifting while increases the velocity in drainage scenario. Also, the frictional parameter and the gravitational parameter have opposite, while material constant has direct relationship with the velocity profile in lifting case. All the parameters showed inverse effect on the velocity in drainage case.

1. Introduction

Thin film flows can be seen in many natural situations such as raindrops on the window, water-filled eyes, and lava. Free drainage refers to a phenomenon in which a fluid flows along a vertical object in such a way that adheres to the form of objects and viscous forces [1]. Paint finishing, oil refining processes, chip production, construction and public works, and laser cutting are industrial applications of these flows [2–4]. The first work on thin films was performed based on Newtonian fluids in [3]. Although this procedure works for a long time, it was not sufficient for the nonlinear analysis of non-Newton liquid such as melted plastics gels, lubricants containing polymeric additives, blood, and foods such as ketchup and honey [5, 6]. Siddiqui et al. address the drainage problems in relation with Phan-Thein-Tanner (PTT) and third grade fluids which flows along an inclined plane in [7, 8]. Siddiqui et al. also analyzed thin film scenario using fourth-grade fluids on vertical cylinders in [9]. Alam et al. [10] investigated thin film of pseudoplastic fluid. Deiber and Cruz analyzed non-Newtonian fluid flow through a circular tube [11]. In terms of flow types, Yih [12] performed the first studies regarding laminar flows in free surface. Landau [13] and Stuart [14] have extended the analysis to the turbulent flows. Nakaya [15] and Lin [16] performed stability analysis taking into account surface tension. Zangooee et al. [17] performed hydrothermal analysis of hybrid nanofluid on a vertical plate with slip effects. Gulzar et al. analyzed magneto-hyperbolic-tangent liquid for different features in [18]. Fallah et al. analyzed nanofluid in a vertical channel taking polynomial boundary [19]. Nayak et al. [20] numerical examine the mixed convection nanofluid over an isothermal thin needle metallic nanomaterial. Ebrahem et al. investigated the significance of Lorentz forces on radiative nanofluid under multiple constraints [21]. Zaher et al. solved boundary layer flow of a non-Newtonian fluid with planktonic microorganism in [22]. Sara et al. analyzed thin blood stream through electroosmotic forces in hybrid nanofluid [23].

In the past few decades, various numerical and homotopy-based techniques have been proposed by many researchers for BVPs [16, 24, 25]. In 1992, Liao proposed homotopy analysis method for BVPs [26, 27]. After that, professor He proposed a combination of homotopy with perturbation for solution of BVPs in [28–30] and has been used successfully to solve many linear and nonlinear [31–33]. Yıldırım [34], Golbabai et al. [35], and Ghasemi et al. [36] solved integro-differential and integral equations through HPM. FDEs have been modelled and studied in signal processing, physics, and biology due to their ability to capture more complex nonlinear phenomena [37–39]. Spasic and Lazarevic discussed the electro viscoelasticity of fractional-order model in [40]. In this continuation, in current paper, we extend the study of thin film flow of fractional third grade fluid in lifting and drainage cases. We formulate the phenomena in the form of fractional differential equations and compute series solutions using homotopy perturbation method (HPM). In the rest of the manuscript, Section 2 is presenting governing equations. Formulation and solution in lifting case are given in Sections 3 and 4. Sections 5 and 6 contain formulation and solution related to drainage case. Results and discussion is in Section 7, while conclusion is given in Section 8.

2. Governing Equations

The fundamental equations are as follows [7, 8]:

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\mathbf{V}\right] = \nabla.\mathbf{T} + \rho\mathbf{b}, \tag{2}$$

where $\mathbf{T}$, $\mathbf{V}$ and ρ are Cauchy stress tensor, velocity and density, respectively. Where

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{3}$$

where $\mathbf{I}$ is the unit tensor, p is the pressure, and $\mathbf{S}$ the extra stress tensor.

$$\mathbf{S} = \left[a + b\left|\sqrt{\frac{1}{2}\operatorname{tr}(\mathbf{A_1})^2}\right|^{n-1}\right]\mathbf{A_1}, \tag{4}$$

where a, b, and n are constants.

$$\begin{aligned} \mathbf{S} + \lambda_1 \frac{\mathrm{D}\mathbf{S}}{\mathrm{D}t} + \frac{\lambda_3}{2}(\mathbf{S}\mathbf{A_1} + \mathbf{A_1}\mathbf{S}) + \frac{\lambda_5}{2}(\operatorname{tr}\mathbf{S})\mathbf{A_1} &= \mu\left(\mathbf{A_1} + \lambda_2\frac{\mathrm{D}\mathbf{A_1}}{\mathrm{D}t} + \lambda_4{\mathbf{A_1}}^2\right), \mathbf{A_1} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}, \\ \mathbf{L} = \operatorname{grad}\ \mathbf{V}, \end{aligned} \tag{5}$$

where $\mu, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are material constant, and $\mathbf{A_1}$ is the Rivlin-Ericksen tensor.

3. Formulation of the Problem in Lifting Case [7, 8]

Substituting Equations (3) and (4) in Equation (2), we get

$$\begin{aligned} -\frac{dp_1}{dx} &= 0, \\ -\frac{dp_1}{dy} + \rho g + a\frac{d^2 v}{dx^2} + b\frac{d}{dx}\left(\frac{dv}{dx}\right)^{\mathrm{n}} &= 0. \end{aligned} \tag{6}$$

From above we deduce that $p_1 = p_1(y)$,

$$a\frac{d^2 v}{dx^2} + nb\left(\frac{dv}{dx}\right)^{\mathrm{n}-1}\frac{d^2 v}{dx^2} + \rho g = \frac{dp_1}{dx}. \tag{7}$$

Equation (9) becomes

$$a\frac{d^2 v}{dx^2} + nb\left(\frac{dv}{dx}\right)^{\mathrm{n}-1}\frac{d^2 v}{dx^2} - \rho \mathrm{g} = 0, \tag{8}$$

$$\text{with } v = U_0 at\, x = 0 \text{ and } S_{xy} = 0 \text{ at } x = \delta, \tag{9}$$

where

$$S_{xy} = a\frac{dv}{dx} + b\left[\frac{dv}{dx}\right]^{\mathrm{n}}. \tag{10}$$

Using Equation (10) in Equation (9), we get

$$\frac{dv}{dx} = 0 \text{ at } x = \delta, \tag{11}$$

$$\frac{d^2 v}{dx^2} + \frac{nb}{a}\left(\frac{dv}{dx}\right)^{\mathrm{n}-1}\frac{d^2 v}{dx^2} - \frac{\rho g}{a} = 0. \tag{12}$$

Substituting $n = 3, b = 2(\beta_2 + \beta_3)$ and $a = \mu$ in Equation (12), we have

$$\frac{d^2 v}{dx^2} + \frac{6(\beta_2 + \beta_3)}{\mu}\left(\frac{dv}{dx}\right)^2\frac{d^2 v}{dx^2} - \frac{\rho g}{\mu} = 0, \tag{13}$$

$$\left.\begin{aligned} v &= U_0 at\, x = 0, \\ \frac{dv}{dx} &= 0 \text{ at } x = \delta. \end{aligned}\right\} \tag{14}$$

$v^* = v/U_0, x^* = x/\delta, \beta^* = 6(\beta_2 + \beta_3)U_0{}^2/\mu, g_p{}^* = \rho g/\mu U_0$ are dimensionless parameters.

The dimensionless form without '*' of Equation (13) subject to Equation (14) is

$$\frac{d^2v}{dx^2} + \beta\left(\frac{dv}{dx}\right)^2 \frac{d^2v}{dx^2} - g_p = 0, \tag{15}$$

$$\text{with } \frac{dv}{dx} = 0 \text{ at } x = 1 \text{ and } v = 1 \text{ at } x = 0, \tag{16}$$

Using definitions of fractional calculus, Equation (16) can be written as fractionally

$$\frac{d^2v(x)}{dx^2} + \beta(D^\alpha v(x))^2 \frac{d^2v(x)}{dx^2} - g_p = 0, \tag{17}$$

$$\text{with } v'(1) = 0, v(0) = 1, 0 < \alpha < 1. \tag{18}$$

4. Homotopy Solution of Third Grade Fluid in Lifting Case

For Equation (17), the homotopy $\Omega \times [0, 1] \longrightarrow R$ is defined as follows [24]:

$$(1-p)\frac{d^2v(x)}{dx^2} + p\left[\frac{d^2v(x)}{dx^2} + \beta(D^\alpha v(x))^2 \frac{d^2v(x)}{dx^2} - g_p\right] = 0. \tag{19}$$

Using Equations (18) and (19) different order problems are given as follows:

0^{th} order

$$v_0{}''(\mathrm{x}) = 1, v_0'(1) = 0, v_0(0) = 1. \tag{20}$$

1^{st} order

$$-g_p + \beta(D^\alpha v_0(\mathrm{x}))^2 v_0{}''(\mathrm{x}) + v_1{}''(\mathrm{x}) = 0, v_1{}'(1) = 0, v_1(0) = 0 \tag{21}$$

2^{nd} order

$$\begin{aligned} &2\beta(D^\alpha v_0(\mathrm{x}))(D^\alpha v_1(\mathrm{x}))v_0{}''(\mathrm{x}) + \beta(D^\alpha v_0(\mathrm{x}))^2 v_1{}''(\mathrm{x}) \\ &\quad + v_2{}''(\mathrm{x}) = 0, v_2{}'(1) = 0, v_2(0) = 0 \end{aligned} \tag{22}$$

3^{rd} order

$$\begin{aligned} &\beta(D^\alpha v_1(\mathrm{x}))^2 v_0{}''(\mathrm{x}) + 2\beta(D^\alpha v_0(\mathrm{x}))(D^\alpha v_2(\mathrm{x}))v_0{}''(\mathrm{x}) \\ &\quad + 2\beta(D^\alpha v_0(\mathrm{x}))(D^\alpha v_1(\mathrm{x}))v_1{}''(\mathrm{x}) \\ &\quad + \beta(D^\alpha v_0(\mathrm{x}))^2 v_2{}''(\mathrm{x}) + v_3{}''(\mathrm{x}) \\ &\quad = 0, v_3(0) = 0, v_3{}'(1) = 0 \end{aligned} \tag{23}$$

4^{th} order

$$\begin{aligned} &2\beta(D^\alpha v_1(\mathrm{x}))(D^\alpha v_2(\mathrm{x}))v_0{}''(\mathrm{x}) + 2\beta(D^\alpha v_0(\mathrm{x}))(D^\alpha v_3(\mathrm{x}))v_0{}''(\mathrm{x}) \\ &\quad + \beta(D^\alpha v_1(\mathrm{x}))^2 v_1{}''(\mathrm{x}) \\ &\quad + 2\beta(D^\alpha v_0(\mathrm{x}))(D^\alpha v_2(\mathrm{x}))v_1{}''(\mathrm{x}) \\ &\quad + 2\beta(D^\alpha v_0(\mathrm{x}))(D^\alpha v_1(\mathrm{x}))v_2{}''(\mathrm{x}) \\ &\quad + \beta(D^\alpha v_0(\mathrm{x}))^2 v_3{}''(\mathrm{x}) \\ &\quad + v_4{}''(\mathrm{x}) = 0, v(0) = 0, v_4{}'(1) = 0 \end{aligned} \tag{24}$$

Using Caputo definition while $\alpha = 0.8$, $\beta = 1$ and $g_p = 0.8$ fixed, the approximate solution is

$$\begin{aligned} V(x) = 1 &+ \frac{1}{2}\left(-1.6x + 0.8x^2\right) \\ &+ \frac{1}{x^{1.6}} 0.01787764010524059\left(7.612661760000037x^{2.6}\right. \\ &- 10.110566399999996x^4 + 6.938624000000001x^5 \\ &- 1.5769600000000001x^6 \end{aligned}$$

$$\text{The residual is } R = \frac{d^2V(x)}{dx^2} + \beta(D^\alpha V(x))^2 \frac{d^2V(x)}{dx^2} - g_p \tag{25}$$

4.1. Flow Rate and Average Velocity in Lifting Case [7].

The average velocity is

$$\begin{gathered} Q = \int_0^1 V(x)dx, \\ Q = \frac{(-3+2\alpha)\left(-3g_p{}^3(-3+\alpha)^2(-16+\alpha(25+\alpha(-13+2\alpha)))\beta/-7+2\alpha-2\left(-3+g_p\right)(-5+2\alpha)\Gamma[4-\alpha]^2\right)}{6(15-16\alpha+4\alpha^2)\Gamma[4-\alpha]^2}. \\ \bar{V} = Q. \end{gathered} \tag{26}$$

Table 1: Results for α in lifting case where $\beta = 0.5$ and $g_p = 0.001$ are fixed.

x	$\alpha = 0.2$		$\alpha = 0.6$		$\alpha = 0.99$	
	$V(x)$	Error	$V(x)$	Error	$V(x)$	Error
0.1	9.9998e−1	-4.54216e−18	9.9991e−1	-1.54976e−17	9.9981e−1	-2.1073e−16
0.2	9.9981e−1	-1.23994e−17	9.9984e−1	-3.1282e−17	9.9985e−1	-1.53593e−16
0.3	9.9974e−1	-2.16777e−17	9.9982e−1	-4.36903e−17	9.9973e−1	-1.01302e−16
0.4	9.9962e−1	-3.07258e−17	9.9972e−1	-5.10537e−17	9.9965e−1	-6.13372e−17
0.5	9.9963e−1	-3.81183e−17	9.9975e−1	-5.31326e−17	9.9959e−1	-3.37629e−17
0.6	9.9958e−1	-4.2865e−17	9.9968e−1	-5.06284e−17	9.9956e−1	-1.65235e−17
0.7	9.9954e−1	-4.45033e−17	9.9964e−1	-4.47699e−17	9.9955e−1	-6.95404e−18
0.8	9.9952e−1	-4.30865e−17	9.9952e−1	-3.69744e−17	9.9954e−1	-2.42827e−18
0.9	9.9998e−1	-4.54216e−18	9.9991e−1	-1.54976e−17	9.9981e−1	-2.1073e−16
1.	9.9981e−1	-1.23994e−17	9.9984e−1	-3.1282e−17	9.9985e−1	-1.53593e−16

Table 2: Results for g_p in lifting case where $\alpha = 0.95$ and $\beta = 0.1$ are fixed.

x	$g_p = 0.001$		$g_p = 0.01$		$g_p = 0.1$	
	$V(x)$	Error	$V(x)$	Error	$V(x)$	Error
0.1	9.9988e−1	-8.42917e−18	9.9985e−1	-8.42917e−13	9.9989e−1	-8.42614e−8
0.2	9.9985e−1	-6.14371e−18	9.9982e−1	-6.14371e−13	9.9986e−1	-6.14184e−8
0.3	9.9982e−1	-4.05208e−18	9.9978e−1	-4.05208e−13	9.9984e−1	-4.05108e−8
0.4	9.9972e−1	-2.45348e−18	9.9975e−1	-2.45348e−13	9.9975e−1	-2.453e−8
0.5	9.9971e−1	-1.35051e−18	9.9973e−1	-1.35051e−13	9.9972e−1	-1.35031e−8
0.6	9.9968e−1	-6.60931e−19	9.9969e−1	-6.60941e−14	9.9968e−1	-6.60868e−9
0.7	9.9964e−1	-2.78159e−19	9.9965e−1	-2.78161e−14	9.9964e−1	-2.78139e−9
0.8	9.9962e−1	-9.71282e−20	9.9964e−1	-9.7131e−15	9.9952e−1	-9.71257e−10
0.9	9.9952e−1	-2.8262e−20	9.9955e−1	-2.82618e−15	9.9951e−1	-2.82607e−10
1.	9.9951e−1	-6.02955e−20	9.992e−1	-6.03165e−16	9.995e−1	-6.03129e−11

Table 3: Results for β in lifting case where $\alpha = 0.99$ and $g_p = 0.001$ are fixed.

x	$\beta = 0.1$		$\beta = 0.5$		$\beta = 0.9$	
	$V(x)$	Error	$V(x)$	Error	$V(x)$	Error
0.1	9.9998e−1	-1.03899e−17	9.9984e−1	-2.59749e−16	9.9979e−1	-8.41587e−16
0.2	9.9991e−1	-6.69572e−18	9.9981e−1	-1.67394e−16	9.9975e−1	-5.42356e−16
0.3	9.9984e−1	-4.02215e−18	9.9974e−1	-1.00554e−16	9.9974e−1	-3.25794e−16
0.4	9.9978e−1	-2.22459e−18	9.9972e−1	-5.56148e−17	9.9962e−1	-1.80192e−16
0.5	9.9973e−1	-1.10445e−18	9.9963e−1	-2.76114e−17	9.9963e−1	-8.9461e−17
0.6	9.9968e−1	-4.71565e−19	9.9961e−1	-1.17891e−17	9.9956e−1	-3.81967e−17
0.7	9.9964e−1	-1.60686e−19	9.9954e−1	-4.01724e−18	9.9954e−1	-1.30159e−17
0.8	9.9962e−1	-3.81773e−20	9.9952e−1	-9.54481e−19	9.9952e−1	-3.09255e−18
0.9	9.9951e−1	-5.30469e−21	9.9951e−1	-1.326e−19	9.995e−1	-4.29646e−19
1.	9.994e−1	-2.27581e−22	9.995e−1	-5.66682e−21	9.994e−1	-1.83058e−20

5. Mathematical Formulation in Drainage Case [7, 8]

Considering the fluid falling on the stationary infinite stationary belt, the flow is in the downward direction due to gravity, so Equation (15) becomes

$$\frac{d^2 v}{dx^2} + \beta \left(\frac{dv}{dx} \right)^2 \frac{d^2 v}{dx^2} + g_p = 0, \tag{27}$$

Table 4: Results for α in drainage case keeping $\beta = 0.1$ and $g_p = 0.001$ are fixed.

x	$\alpha = 0.2$ $V(x)$	Error	$\alpha = 0.6$ $V(x)$	Error	$\alpha = 0.99$ $V(x)$	Error
0.1	9.41e−5	1.81686e−19	9.52e−5	2.04663e−18	9.62e−5	1.03899e−17
0.2	1.71e−4	4.95977e−19	1.73e−4	2.82993e−18	1.81e−4	6.69572e−18
0.3	2.62e−4	8.67109e−19	2.65e−4	2.98548e−18	2.64e−4	4.02215e−18
0.4	3.42e−4	1.22903e−18	3.43e−4	2.74607e−18	3.53e−4	2.22459e−18
0.5	3.65e−4	1.52473e−18	3.62e−4	2.29456e−18	3.65e−4	1.10445e−18
0.6	4.12e−4	1.7146e−18	4.11e−4	1.76941e−18	4.52e−4	4.71565e−19
0.7	4.52e−4	1.78013e−18	4.54e−4	1.26636e−18	4.61e−4	1.60686e−19
0.8	4.91e−4	1.72346e−18	4.93e−4	8.42172e−19	4.89e−4	3.81773e−20
0.9	4.95e−4	1.56378e−18	4.91e−4	5.20346e−19	4.95e−4[4]	5.30469e−21
1.	5.11e−4	1.33187e−18	5.21e−4	2.98346e−19	5.01e−4	2.27581e−22

Table 5: Results for g_p in drainage case where $\alpha = 0.99$ and $\beta = 0.1$ are fixed.

x	$g_p = 0.1$ $V(x)$	Error	$g_p = 0.01$ $V(x)$	Error	$g_p = 0.001$ $V(x)$	Error
0.1	9.41e−5	1.81686e−19	9.52e−5	2.04663e−18	9.62e−5	1.03899e−17
0.2	1.71e−4	4.95977e−19	1.73e−4	2.82993e−18	1.81e−4	6.69572e−18
0.3	2.62e−4	8.67109e−19	2.65e−4	2.98548e−18	2.64e−4	4.02215e−18
0.4	3.42e−4	1.22903e−18	3.43e−4	2.74607e−18	3.53e−4	2.22459e−18
0.5	3.65e−4	1.52473e−18	3.62e−4	2.29456e−18	3.65e−4	1.10445e−18
0.6	4.12e−4	1.7146e−18	4.11e−4	1.76941e−18	4.52e−4	4.71565e−19
0.7	4.52e−4	1.78013e−18	4.54e−4	1.26636e−18	4.61e−4	1.60686e−19
0.8	4.91e−4	1.72346e−18	4.93e−4	8.42172e−19	4.89e−4	3.81773e−20
0.9	4.95e−4	1.56378e−18	4.91e−4	5.20346e−19	4.95e−4[4]	5.30469e−21
1.	5.11e−4	1.33187e−18	5.21e−4	2.98346e−19	5.01e−4	2.27581e−22

Table 6: Results for β in drainage case where $\alpha = 0.95$ and $g_p = 0.001$ are fixed.

x	$\beta = 0.1$ $V(x)$	Error	$\beta = 0.3$ $V(x)$	Error	$\beta = 0.7$ $V(x)$	Error
0.1	9.41e−5	1.81686e−19	9.52e−5	2.04663e−18	9.62e−5	1.03899e−17
0.2	1.71e−4	4.95977e−19	1.73e−4	2.82993e−18	1.81e−4	6.69572e−18
0.3	2.62e−4	8.67109e−19	2.65e−4	2.98548e−18	2.64e−4	4.02215e−18
0.4	3.42e−4	1.22903e−18	3.43e−4	2.74607e−18	3.53e−4	2.22459e−18
0.5	3.65e−4	1.52473e−18	3.62e−4	2.29456e−18	3.65e−4	1.10445e−18
0.6	4.12e−4	1.7146e−18	4.11e−4	1.76941e−18	4.52e−4	4.71565e−19
0.7	4.52e−4	1.78013e−18	4.54e−4	1.26636e−18	4.61e−4	1.60686e−19
0.8	4.91e−4	1.72346e−18	4.93e−4	8.42172e−19	4.89e−4	3.81773e−20
0.9	4.95e−4	1.56378e−18	4.91e−4	5.20346e−19	4.95e−4	5.30469e−21
1.	5.11e−4	1.33187e−18	5.21e−4	2.98346e−19	5.01e−4	2.27581e−22

$$\text{with} \left.\begin{array}{l} v = 0 \text{ at } x = 0, \\ \dfrac{dv}{dx} = 0 \text{ at } x = 1. \end{array}\right\} \quad (28)$$

Using definitions of fractional calculus, Equation (27) can be written fractionally as follows:

$$\frac{d^2 v(x)}{dx^2} + \beta (D^\alpha v(x))^2 \frac{d^2 v(x)}{dx^2} + g_p = 0, \quad (29)$$

$$\text{with } v(0) = 0, v'(1) = 0, 0 < \alpha < 1. \quad (30)$$

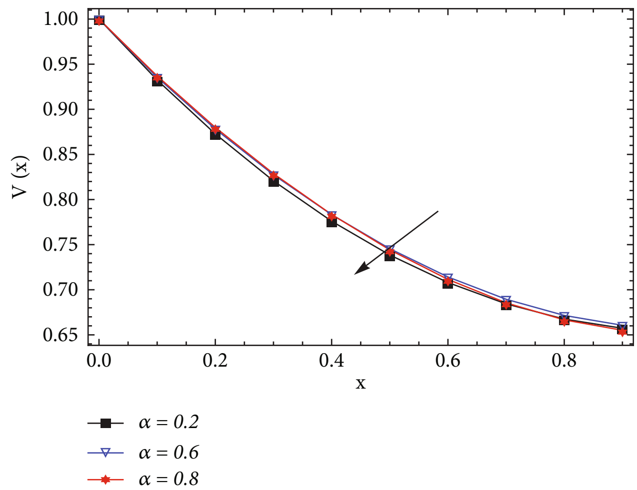

FIGURE 1: In lifting case effect of α on $V(x)$ where $g_p = 0.8$ and $\beta = 1$ are fixed.

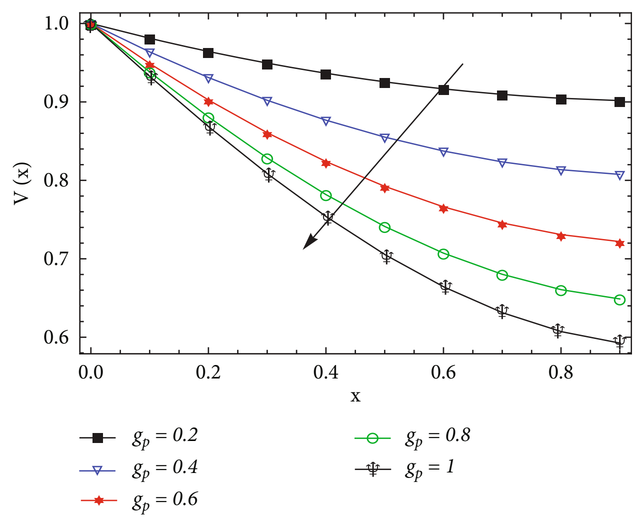

FIGURE 2: In lifting case effect of g_p on $V(x)$ where $\alpha = 0.95$ and $\beta = 1$ are fixed.

6. Homotopy Solution of Third Grade Fluid in Drainage Case

For Equation (29), the homotopy $\Omega \times [0, 1] \longrightarrow R$ is defined as follows [24]:

$$(1-p)\frac{d^2 v(x)}{dx^2} + p\left[\frac{d^2 v(x)}{dx^2} + \beta(D^\alpha v(x))^2 \frac{d^2 v(x)}{dx^2} + g_p\right] = 0. \tag{31}$$

Using Equations (30) and (31) different order problems are given as follows:

0^{th} order

$$v_0''(x) = 0, v_0'(1) = 0, v_0(0) = 0. \tag{32}$$

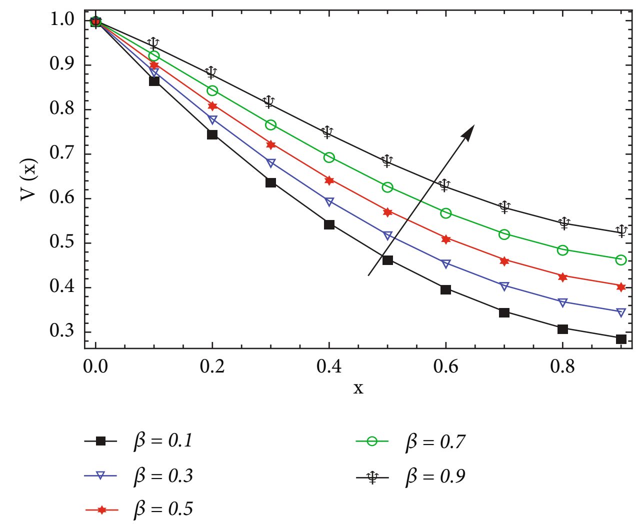

FIGURE 3: In lifting case effect of β on $V(x)$ where $\alpha = 0.95$ and $g_p = 1.5$ are fixed.

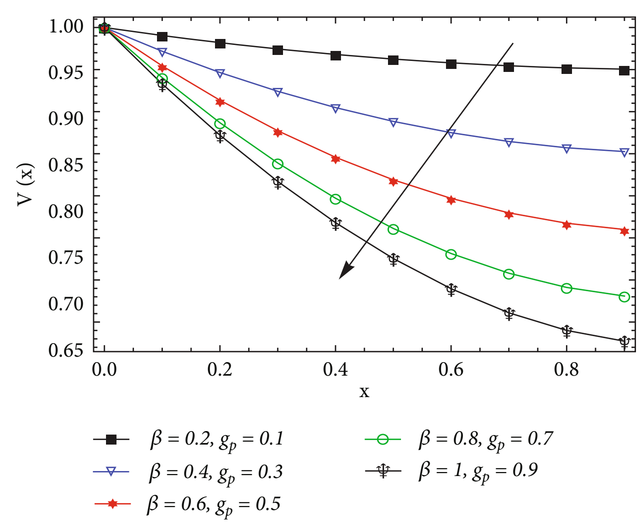

FIGURE 4: In lifting case effect of increasing β and g_p on $V(x)$ where $\alpha = 0.8$ is fixed.

1^{st} order

$$g_p + \beta(D^\alpha v_0(x))^2 v_0''(x) + v_1''(x) = 0, v_1'(1) = 0, v_1(0) = 0. \tag{33}$$

2^{nd} order

$$\begin{aligned} 2\beta(D^\alpha v_0(x))(D^\alpha v_1(x))v_0''(x) + \beta(D^\alpha v_0(x))^2 v_1''(x) \\ + v_2''(x) = 0, v_2(0) = 0, v_2'(1) = 0. \end{aligned} \tag{34}$$

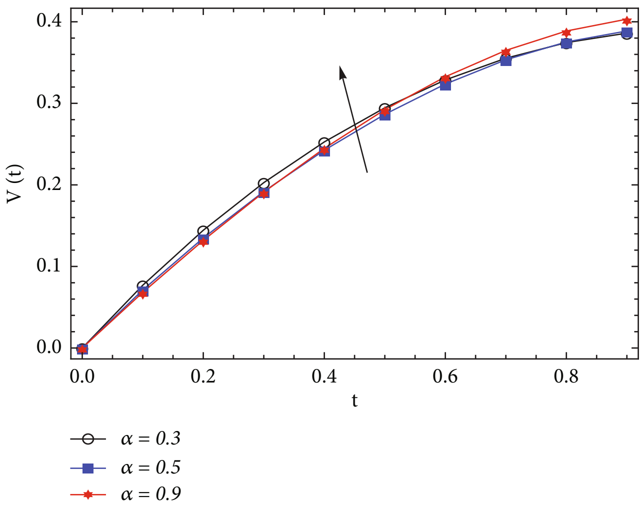

FIGURE 5: In drainage case effect of α on $V(x)$ where $g_p = 1$ and $\beta = 1$ are fixed.

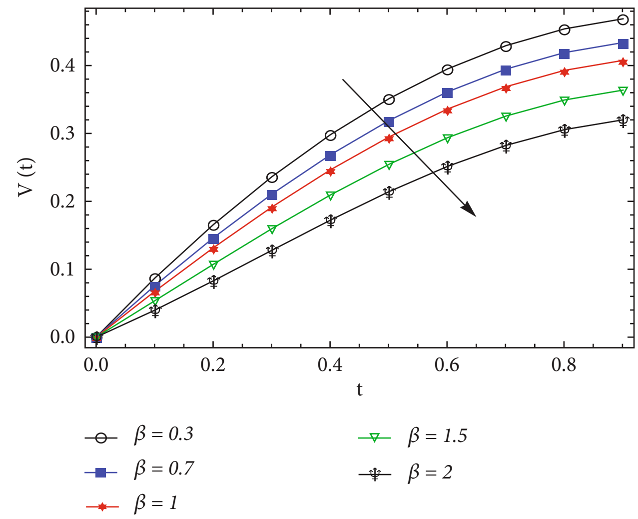

FIGURE 7: In drainage case effect of β on $V(x)$ where $g_p = 1$ and $\alpha = 0.95$ are fixed.

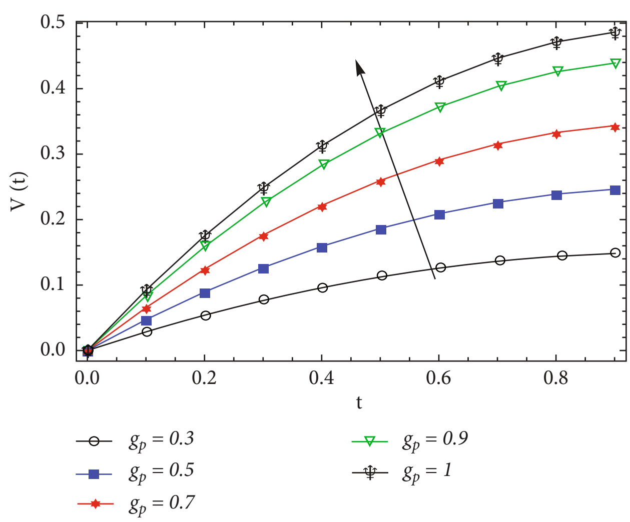

FIGURE 6: In drainage case effect of g_p on $V(x)$ keeping $\alpha = 0.95$ and $\beta = 1$ are fixed.

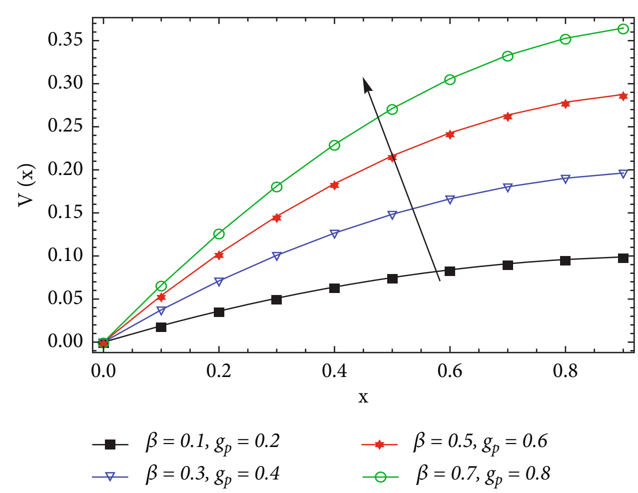

FIGURE 8: In drainage case effect of increasing β and g_p on $V(x)$ where $\alpha = 0.95$ is fixed.

3rd order

$$\begin{aligned}\beta(D^{\alpha}v_1(\mathrm{x}))^2 v_0{}''(\mathrm{x}) &+ 2\beta(D^{\alpha}v_0(\mathrm{x}))(D^{\alpha}v_2(\mathrm{x}))v_0{}''(\mathrm{x})\\ &+ 2\beta(D^{\alpha}v_0(\mathrm{x}))(D^{\alpha}v_1(\mathrm{x}))v_1{}''(\mathrm{x})\\ &+ \beta(D^{\alpha}v_0(\mathrm{x}))^2 v_2{}''(\mathrm{x}) + v_3{}''(\mathrm{x})\\ &= 0, v_3(0) = 0, v_3{}'(1) = 0.\end{aligned} \tag{35}$$

4th order

$$\begin{aligned}2\beta(D^{\alpha}v_1(\mathrm{x}))(D^{\alpha}v_2(\mathrm{x}))v_0{}''(\mathrm{x}) &+ 2\beta(D^{\alpha}v_0(\mathrm{x}))(D^{\alpha}v_3(\mathrm{x}))v_0{}''(\mathrm{x})\\ &+ \beta(D^{\alpha}v_1(\mathrm{x}))^2 v_1{}''(\mathrm{x})\\ &+ 2\beta(D^{\alpha}v_0(\mathrm{x}))(D^{\alpha}v_2(\mathrm{x}))v_1{}''(\mathrm{x})\\ &+ 2\beta(D^{\alpha}v_0(\mathrm{x}))(D^{\alpha}v_1(\mathrm{x}))v_2{}''(\mathrm{x})\\ &+ \beta(D^{\alpha}v_0(\mathrm{x}))^2 v_3{}''(\mathrm{x})\\ &+ v_4{}''(\mathrm{x}) = 0, v(0) = 0, v_4{}'(1) = 0.\end{aligned} \tag{36}$$

By using definition where $\alpha = 0.8, \beta = 1$ and $g_p = 0.8$ are fixed, we get the following approximate solution:

$$V(x) = 1/2\left(0.002\,x - 0.001\,x^2\right) + \frac{\left(0.0269608\left(-1.08662\times 10^{-9}\,x^{2.8} + 1.55232\times 10^{-9}\,x^4 - 1.0584\times 10^{-9}x^5 + 2.52\times 10^{-10}x^6\right)\right)}{x^{1.8}}.$$

$$\text{The residual is } R = \frac{d^2V(x)}{dx^2} + \beta(D^\alpha V(x))^2\frac{d^2V(x)}{dx^2} + g_p. \tag{37}$$

6.1. Average Velocity and Flow Rate in Drainage Case. The average velocity is

$$Q = \int_0^1 V(x)dx,$$

$$Q = \frac{g_p(-3+2\alpha)\left(3g_p{}^2(-3+\alpha)^2(-16+\alpha(25+\alpha(-13+2\alpha)))\beta + (70+8(-6+\alpha)\alpha)\Gamma[4-\alpha]^2\right)}{2(-21+6\alpha)(15-16\alpha+4\alpha^2)\Gamma[4-\alpha]^2}.\bar{V} = Q. \tag{38}$$

7. Result and Discussion

In this article, series solution of fractional thin film of third grade fluid is obtained in case of lifting and drainage. For the validity check, modelled problems are solved for different values of involved parameters, and the results are presented in Tables 1–6. Tables 1 and 4 are showing solutions and residual errors for different values of fractional parameter α. Tables 2 and 5 present solution and errors against different numerical values of gravitational parameter g_p. Similarly, Tables 3 and 6 show the solutions along with errors for different values of non-Newtonian parameter β. Analysis of these tables clearly indicates that obtained solutions are valid and consistent. Graphical analysis of the involved parameters is provided in Figures 1–8. Figures 1–4 capture the effect of involved parameter on the velocity in lifting case. Figures 1, 2, and 3 show the effects of fractional, gravitational, and material parameter on the velocity profile. It is observed that α and g_p have inverse, while β has direct relationship with the fluid velocity in lifting case. The effect of simultaneous increase in β and g_p on the velocity is shown in Figure 4. It has been observed that g_p effect is more dominant as compared to β in case of lifting. Effects of above mentioned parameters in drainage case are shown in Figures 5–8. Figures 5, 6, and 7 fractional, gravitational, and material parameter on the velocity profile. It is observed that α and g_p have direct while β has inverse relationship with the fluid velocity in drainage case. The effect of simultaneous increase in β and g_p on the velocity is shown in Figure 8. It has been observed that the effect of g_p is more dominant as compared to β in drainage as well.

8. Conclusions

In this article, homotopy based solutions of fractional thin film of third grade fluid are obtained. The validity and convergence of the obtained solutions are confirmed by finding residual errors in each case. The effects of different parameters (fluid and fractional) are also explored on the fluid velocity in fractional environment. Analysis reveals that fractional parameter showed inverse behavior on the fluid velocity in lifting and drainage cases. Moreover, gravitational parameter is prevailing parameter as compared to other fluid parameters in this study.

Nomenclature

T: Cauchy stress tensor
V: Velocity vector
ρ: Density
S: Extra stress tensor
$\mathbf{A}_{1,}$: Rivlin-Ericksen tensor
β: Non-Newtonian Parameter
g: Gravitational force
μ: Material constant
λ_i: Material constants
g_p: gravitational parameter
β_i: Material constants
α: Fractional parameter.

Data Availability

All the data is available with in the manuscript.

Conflicts of Interest

Authors have no conflict of interest on the publication of this article.

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