Chaos Control of a Resource-Economic-Pollution Dynamic System

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

How to control chaos in the economic system has aroused the interest of researchers. We research the chaos control in a new Resource-Economic-Pollution system by time-delayed feedback control. By determining the appropriate range of time delay τ and feedback strength k, the chaotic phenomena of the system are controlled. We verify the linear stability and the existence of Hopf bifurcation of the system. Numerical simulations show that chaos control can eliminate the chaotic behavior of the system and stabilize the system at the equilibrium point. When the time lag term is in a certain interval, the chaotic phenomenon of the system will disappear and the system will be controlled in a stable state. In practice, due to capacity and financial constraints, the firm or the government often restrains output through many methods to confine the range of fluctuations in these variables. This shows that the government or corporate decision makers have often used this approach consciously or unconsciously to promote steady economic growth.

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1 INTRODUCTION

The influence of chaos on modern science is not limited to the natural sciences, but also involves economics, sociology, biology, and so on [1, 2, 3]. However, the appearance of chaos often leads to abnormal behavior of the system. Economic systems are nonlinear in nature. In general, chaotic phenomena must arise from nonlinear systems[4]. Chaos may lead to economic instability, economic crisis in serious cases, and even threaten the security and stability of the country. Thence how to control the chaos has caused widespread concern among scholars. [5, 6, 7, 8].

Recently, researchers have become increasingly interested in controlling the chaos in economic systems by different methods [9, 10, 11, 12, 13]. Ott, Grebogi and Yorke [14] proposed the OGY control method, which was successfully applied to chaos control in some economic systems [15, 16]. Kass linked the chaotic target method with the OGY method to stabilize chaos in dynamic macroeconomic models [17]. These methods need accurate system information prior to implementation. Therefore, to make an accurate decision, the government must have a large amount of relevant economic data, which is impractical or very expensive [18]. Pyragas [19] proposed a method for chaotic control using delayed feedback signals. A lot of researches have been carried out on time-delay control [20, 21, 22, 23, 24, 25]. Compared with other methods, time-delayed control is a simple and effective method to control chaos in economic models. Because it does not require any system information [26]. Thus, we adopt the time-delayed feedback control method in the present paper.

In this paper, we use the time-delayed feedback control to research the REP system [27]. Because unstable fluctuations are always regarded as unfavorable phenomena in traditional economics [18], we added time delays

for new economy scale. Under appropriate feedback strength, the economic scale chaotic system is controlled to a steady state. The local stability of the system and the existence of Hopf bifurcation are studied by theoretical analysis. It is proved that the chaos disappears when the time delay reaches a certain value.

This paper is organized as follows. In Section 2, based on the delayed feedback control method, the stability and the period of one of the equilibrium points are analyzed. To verify the theoretical analysis, Section 3 gives numerical simulations. Finally, a conclusion is given in Section 4.

2 ESTABLISHMENT OF THE MODEL

Yin et al. discovered a novel chaotic system:

$$
\begin{cases}
\dfrac{dx}{dt} = a_1 x + a_2 y - a_3 yz - a_4 x, \\
\dfrac{dy}{dt} = b_1 x(1 - \dfrac{x}{M}) - b_2 y - b_3 z + b_4(z - x), \\
\dfrac{dz}{dt} = c_1 xy - c_2 z + c_3(y - x),
\end{cases} \quad (2.1)
$$

where $x(t)$ is the total resource consumed in a region during a given period, $y(t)$, of economy scale, $z(t)$, of pollution [27]. a_i, b_i, c_i, (i=1,...,3) are positive system parameters, and M represents the maximum value of resource consumption.

We control the chaotic system (2.1) with the time-delayed feedback strategy. We apply the time-delayed force $k[y(t) - y(t - \tau)]$ to the second equation of the system (2.1), where τ is a time-delay term, satisfying τ greater than 0 and $k \in R$. The magnitude of the τ value is the length of the reaction time, k is the feedback gain. Since the output in the previous period will affect the current economic growth rate, the control term $k[y(t) - y(t - \tau)]$ is introduced, where $k[y(t) - y(t - \tau)]$ represents the production within a certain period of time. Then the system (2.1) is described as:

$$
\begin{cases}
\dot{x} = a_1 x + a_2 y - a_3 yz - a_4 x, \\
\dot{y} = b_1 x(1 - \dfrac{x}{M}) - b_2 y - b_3 z + b_4(z - x) + k[y(t) - y(t - \tau)], \\
\dot{z} = c_1 xy - c_2 z + c_3(y - x),
\end{cases} \quad (2.2)
$$

By the linear transform, system (2.2) is changed to:

$$\begin{cases} \dot{x} = a_{11}x + a_{12}y + a_{13}z, \\ \dot{y} = a_{21}x + a_{22}y + a_{23}z + a_{24}y(t-\tau), \\ \dot{z} = a_{31}x + a_{32}y + a_{33}z, \end{cases} \quad (2.3)$$

where

$$a_{11} = a_1 - a_4, a_{12} = a_2 - a_3 z^*, a_{13} = -a_3 y^*, a_{21} = b_1 - b_4 - 2b_1 x^*/M,$$
$$a_{22} = -b_2 + k, a_{23} = b_4 - b_3, a_{24} = -k, a_{31} = c_1 y^* - c_3, a_{32} = c_1 x^* + c_3, a_{33} = -c_2.$$

The characteristic equation system (2.3) is

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau} = 0, \quad (2.4)$$

where $A_2 = -a_{11} - a_{22} - a_{33}$, $A_1 = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32}$, $A_0 = a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}$ $B_2 = -a_{24}$, $B_1 = a_{11}a_{24} + a_{24}a_{33}$, $B_0 = a_{13}a_{24}a_{31} - a_{11}a_{24}a_{33}$. In order to analyze the distribution of roots of the transcendental equation (2.4), we introduce the following lemma.

Lemma 1. *[28] Consider the transcendental equation:*

$$P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} +$$
$$\cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m} = 0, \quad (2.5)$$

where $\tau_j > 0(j = 1, 2, \cdots, m)$ and $p_k^{(j)}(j = 0, 1, 2, \cdots, m; k = 1, 2, \cdots, n)$ are constants. As $(\tau_1, \tau_2, \cdots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ on the open right half plane can change, and only a zero appears on or crosses the imaginary axis.

When $\tau = 0$, Eq.(2.4) has the form :

$$\lambda^3 + (A_2 + B_2)\lambda^2 + (A_1 + B_1)\lambda + A_0 + B_0 = 0. \quad (2.6)$$

By the Routh-Hurwitz criterion, all the roots of Eq.(2.6) have negative real parts if and only if:

$$A_2 + B_2 > 0, A_0 + B_0 > 0, (A_2 + B_2)(A_1 + B_1) > A_0 + B_0, \quad (H1)$$

Therefore, the equilibrium point is stable when the condition $(H1)$ is satisfied.

According to the Hopf bifurcation theory [29], let $\lambda = \pm i * \omega$ be a root of the Eq. (2.4). Then we can obtain:

$$-\omega^3 i - A_2\omega^2 + A_1\omega i + A_0 + (-B_2\omega^2 + B_1\omega i + B_0)e^{-\omega\tau i} = 0. \quad (2.7)$$

Separating the real and imaginary parts, we have

$$\begin{cases} (B_0 - B_2\omega^2)cos\omega\tau + B_1\omega sin\omega\tau = A_2\omega^2 - A_0, \\ B_1\omega cos\omega\tau - (B_0 - B_2\omega^2)sin\omega\tau = \omega^3 - A_1\omega. \end{cases} \quad (2.8)$$

Adding the squares of both sides, we get:

$$(B_0 - B_2\omega^2)^2 + (B_1\omega)^2 = (A_2\omega^2 - A_0)^2 + (\omega^3 - A_1\omega)^2,$$

which is equivalent to

$$\omega^6 + p\omega^4 + q\omega^2 + r = 0, \tag{2.9}$$

where

$$p = A_2^2 - B_2^2 - 2A_1, q = A_1^2 - 2A_0A_2 + 2B_0B_2 - B_1^2, r = A_0^2 - B_0^2.$$

Denoting $z = \omega^2$, then Eq.(2.9) becomes

$$z^3 + pz^2 + qz + r = 0. \tag{2.10}$$

Let

$$h(z) = z^3 + pz^2 + qz + r. \tag{2.11}$$

Since $\lim\limits_{t \to +\infty} h(z) = +\infty$ and $h(0) = r = A_0^2 - B_0^2$, we assume that

$$\Delta = p^2 - 3q > 0, z^* = \frac{-p \pm \sqrt{\Delta}}{3} > 0, h(z^*) \le 0. \tag{H2}$$

Suppose that Eq. (2.11) has two positive roots z_1 and z_2. Then Eq. (2.9) has two positive roots $\omega_k = \sqrt{z_k}, k = 1, 2$. The corresponding critical value of time delay τ_k^j is

$$\tau_k^{(j)} = \frac{1}{\omega_k}\{arccos[\frac{(A_2\omega_k^2 - A_0)(B_0 - B_2\omega_k^2) + B_1\omega_k(\omega_k^3 - A_1\omega_k)}{(B_0 - B_2\omega_k^2)^2 + (B_1\omega_k)^2}] + 2j\pi\}, \tag{2.12}$$

where $k = 1, 2; j = 0, 1, 2, \cdots$.

Lemma 2. *[29] If the conditions $(H1)$ and $(H2)$ hold, the following cross-sectional conditions are available.*

$$[\frac{d(Re\lambda(\tau))}{d\tau}] \ne 0. \tag{2.13}$$

Let $\lambda(\tau)$ into Eq.(2.4) and derive τ, it follows that

$$
\begin{aligned}
[\frac{d\lambda}{d\tau}]^{-1} &= \frac{3\lambda^2 + 2A_2\lambda + A_1 + (2B_2\lambda + B_1)e^{-\lambda\tau}}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\
&= \frac{(3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau} + (2B_2\lambda + B_1)}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)} - \frac{\tau}{\lambda}.
\end{aligned} \tag{2.14}
$$

Then

$$
\begin{aligned}
[\frac{d(Re\lambda(\tau))}{d\tau}]^{-1}_{\tau=\tau_k^{(j)}} &= Re\{\frac{(3\lambda^2 + 2A_2\lambda + A_1)e^{\lambda\tau} + (2B_2\lambda + B_1)}{\lambda(B_2\lambda^2 + B_1\lambda + B_0)}\}_{\tau=\tau_k^{(j)}} \\
&= Re\{\frac{(-3\lambda_k^2 + 2A_2\lambda_k i + A_1)(cos\omega_k\tau_k^{(j)} + isin\omega_k\tau_k^{(j)}) + 2B_2\lambda_k i + B_1}{\lambda_k i(-B_2\lambda_k^2 + B_1\lambda_k i + B_0)}\} \\
&= Re\{\frac{M_1 + M_2 i}{N_1 + N_2 i}\} = \frac{M_1N_1 + M_2N_2}{N_1^2 + N_2^2},
\end{aligned} \tag{2.15}
$$

where

$$M_1 = (A_1 - 3\omega_k^2)cos\omega_k\tau_k^{(j)} - 2A_2\omega_k sin\omega_k\tau_k^{(j)} + B_1,$$
$$M_2 = 2A_2\omega_k cos\omega_k\tau_k^{(j)} + (A_1 - 3\omega_k^2)sin\omega_k\tau_k^{(j)} + 2B_2\omega_k,$$
$$N_1 = -B_1\omega_k^2, N_2 = \omega_k(A_0 - B_2\omega_k^2).$$

Assume that the following condition holds

$$M_1N_1 + M_2N_2 \neq 0. \qquad (H3)$$

Based on the above analysis, we obtain the following theorem.

Theorem 1. *For system(2.2), then Eq.(2.9) has two positive roots ω_1^0 and ω_2^0, the corresponding critical values of time delay are τ_1^0 τ_2^0.*
(1) If $\tau\epsilon(\tau_1^0, \tau_2^0)$, the equilibrium S is asymptotically stable.
(2) If $\tau\epsilon[0, \tau_1^0)\bigcup(\tau_2^0, +\infty)$, the equilibrium S is unstable. Furthermore, when $\tau = \tau_1^0, \tau_2^0$, the system(2.2) undergoes a Hopf bifurcation at the equilibrium S.

3   DYNAMIC ANALYSIS OF THE MODEL

In this section, to verify and demonstrate the effectiveness and the feasibility of the presented control method, the simulation results have been performed. When $a_1 = 0.065, a_2 = 0.035, a_3 = 0.065, a_4 = 0.04, b_1 = 0.5, b_2 = 0.088, b_3 = 0.06, b_4 = 0.07, c_1 = 0.468, c_2 = 0.06, c_3 = 0.001$, and $M = 10$, system (2.1) has five equilibrium points: E_0, E_1, E_3, E_4 and E_5. We just study the stability of the equilibrium point $E_1(0.1383, 0.6807, 0.7388)$, the other four equilibrium points can be similarly discussed. We take $k = -0.1$. Then we consider the following system:

$$\begin{cases} \dot{x} = 0.025x + 0.35y - 0.065yz \\ \dot{y} = 0.5x(1 - x/10) - 0.088y + 0.01z - 0.07x - 0.1[y(t) - y(t-\tau)] \\ \dot{z} = 0.468xy - 0.06z + 0.001(y - x) \end{cases} \qquad (3.1)$$

It is not difficult to verify that the conditions $(H1)$-$(H3)$ hold. For $j = 0$, we can get $\omega_1^0 = 0.25634, \tau_1^0 = 8.14583; \omega_2^0 = 0.14392, \tau_2^0 = 33.17992$.

From Theorem 1, we know that the system is still chaotic when the delay $\tau = 0$. Fig 1. shows the chaotic attractor.

When τ is changed from 0 to τ_1^0, the chaotic attractor of the system disappears and the period solution appears (Fig 2). Chaos first is changed to limit cycle when $\tau = 2$ (Fig 2(a)), and then the limit cycle becomes a cycle when $\tau = 4$ (Fig 2(b)).

The equilibrium point S is stable for $\tau\epsilon(\tau_1^0, \tau_2^0)$, which is illustrated in Fig 3. Fig 3 shows the stability of the equilibrium point S with time delay for the values of the parameters. When the values of parameters are $\tau = 14$, $\tau = 25$, Fig 3(a) and Fig 3(b) shows the stability of the equilibrium point S.

The numerical simulations show that when $\tau > \tau_2^0$, the bifurcating periodic solution disappear gradually and chaos occurs again finally in Fig 4. As is shown in the picture, when $\tau = 46$ the system is periodic (Fig 4(a)), and when $\tau = 58$ the system 3.1 loses its stability and a Hopf bifurcation occurs (Fig 4(b)).
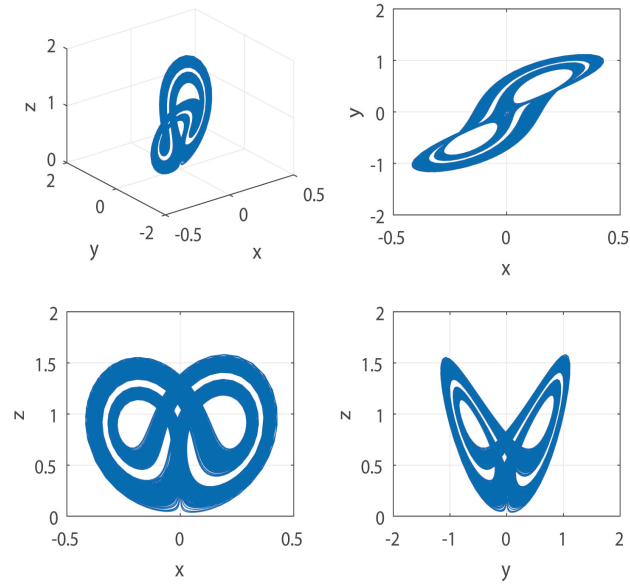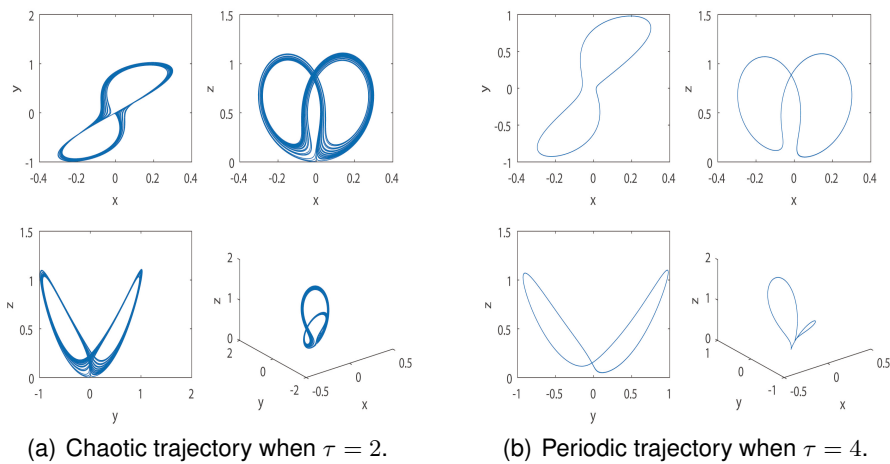
Fig. 1. A chaotic attractor of the REP system for $\tau = 0$**.**

(a) Chaotic trajectory when $\tau = 2$. (b) Periodic trajectory when $\tau = 4$.

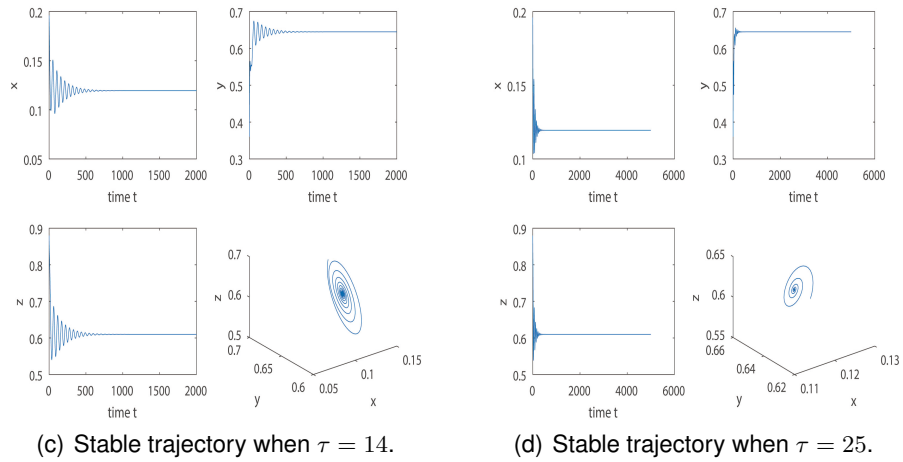**Fig. 2. Phase portrait of system (3.1) in 2-D and 3-D spaces for different time delays.**

(c) Stable trajectory when $\tau = 14$.   (d) Stable trajectory when $\tau = 25$.

Fig. 3. Time series and 3-D phase portrait of system (3.1) for different time delays.

(e) Periodic trajectory when $\tau = 46$.   (f) Bifurcating periodic trajectory when $\tau = 58$.
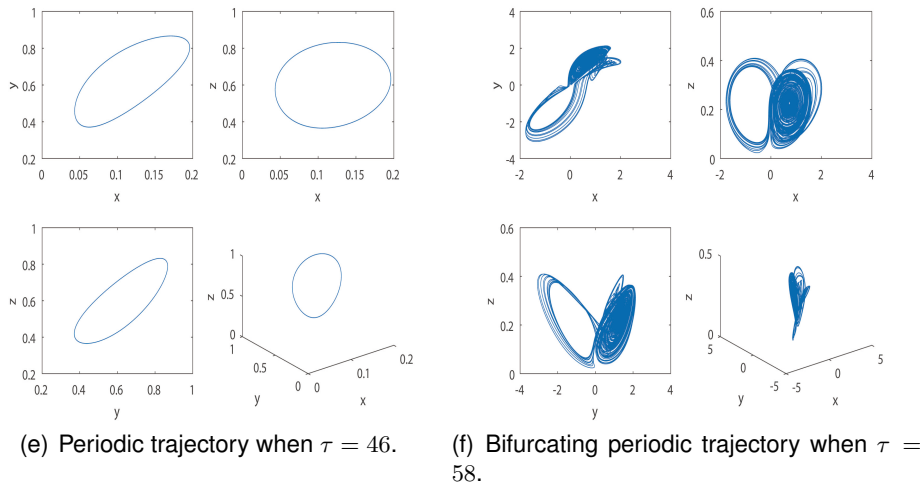
Fig. 4. Phase portrait of system (3.1) in 2-D and 3-D spaces for different time delays.

Therefore, the government should consider the impact of output in the previous period on the current economic growth rate, and make corresponding adjustments. The appropriate intervention and regulation by the government can promote the steady development of the economy to a certain extent.

4   CONCLUSIONS

In this paper, we study the new REP system at hopf bifurcation occurs and the stability of equilibrium. We determine the appropriate range of time delay τ and feedback strength k through some theorems. Our theoretical results and numerical simulations show that the time delay can control the chaotic phenomenon of the system (2.1). As the delay further increases, numerical simulations show that the periodic solution disappears and the chaotic attractor reappears. In real life, the government can take appropriate measures to regulate the economy and stabilize the economy.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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