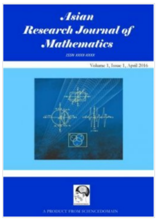

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On Quasiconvex Functions on Time Scales

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Authors' contributions

Author GA drafted the first manuscript. Author MMI played a role in the proofs of some of the theorems and editing. Both authors read and approved the final draft.

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Short Communication

Abstract

The notion of quasiconvex functions on time scales is presented. Some properties such as set relations, inequalities, continuity and differentiability of quasiconvex functions on time scales are established.

Keywords: Time scales; quasiconvex functions; convex functions; semicontinuity.

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1 Introduction

Convex functions have wide applications in Mathematical analysis and also play significant role in our everyday life through the applications in industry, business, medicine etc. The theory of

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quasiconvex function is part of the general subject of convex analysis. This paper however presents the notion of quasiconvex functions on time scales with some key properties.

The concept of time scales was introduced by Stefan Hilger in 1988 in his PhD thesis in order to hybridize continuous and discrete analysis. The study on time scales exposes such discrepancies and helps in the understanding of the difference between the cases. Thus, time scale calculus is a very important tool in many computational and numerical analysis and has applications in biology, mathematical finance, probability theory, population dynamics, etc. For more detail discussion on the calculus of time scales see [1], [2], [3], [4], [5] and [6].

[7] established some results on convex functions on time scales and this paper seeks to extend those results to deal with quasiconvex functions. For further details on convex functions, refer to [8] and [9]. In [10], a review of quasiconvex functions is also presented in a condensed form providing refinements. The paper also established the structure underpining quasiconvex functions and presented some analogues to the properties of convex functions.

2 Materials and Methods

We begin this section with some basic concepts and definitions. A time scale (which is a special case of a measure chain) is an arbitrary non empty closed subset of real numbers (together with the topology of subspace of $\mathbb{R}$. The set of real numbers ($\mathbb{R}$), the set of integers ($\mathbb{Z}$),the set of natural numbers ($\mathbb{N}$) and the set of non-negative integers ($\mathbb{N}_0$) are examples of time scales as well as $[0,1]\cup[2,3], [0,1]\cup\mathbb{N}$, and the cantor set. The set of rational numbers ($\mathbf{Q}$), the set of irrational numbers ($\mathbb{R}\setminus\mathbf{Q}$), the set of complex numbers ($\mathbf{C}$) and the open interval between 0 and 1 [i.e. (1,0)] are not time scales. [5], [11].

Introducing the delta derivative f^{Δ} for a function f define on time scales $\mathbb{T}$, we have [5]

(i) $f^{\Delta}=f'$ is the used derivative if $\mathbb{T}$=$\mathbb{R}$ and
(ii) $f^{\Delta}=\Delta f$ is the forward difference operator if $\mathbb{T}$=$\mathbb{Z}$.

Here, we introduce the basic notions connected to time scales and differentiability of functions on them and consider the above two cases as examples. The general theory is applicable to many more time scales $\mathbb{T}$. Let us first define the forward and the backward jump operators and other related terms as found in [5], [7] and [12].

Definition 2.1. Let $\mathbb{T}$ be a time scale. For $t\in\mathbb{T}$, the mapping $\sigma,\rho:\mathbb{T}\to\mathbb{T}$, such that $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ and $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$ are called the forward and backward jump operators respectively. The convention in this instance is:

$$\inf(\phi)=\sup\mathbb{T}[\sigma(t)=t \quad \text{if} \quad \mathbb{T} \text{ has a maximum t }]$$

and

$$\sup(\phi)=\inf\mathbb{T}[\quad \text{i.e,} \quad \rho(t)=t \quad \text{if} \quad \mathbb{T} \text{ has a minimum t }],$$

where ϕ denotes the null set if $\sigma(t)>t$ and we say that t is right-scattered, while if $\phi(t)<t$ we say that t is left-scattered. Points that are right scattered at the same time left scattered are called isolated. Also, if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then t is called right dense and if $t>\inf\mathbb{T}$ and $\rho(t)=t$, then t is called left dense. Points that are right-dense and left-dense at the same time are called dense.

These jump operators enable us to classify points $\{t\}$ of a time scale as right-dense and left-scattered depending on whether $\sigma(t)=t$, $\sigma(t)>t$, $\rho(t)=t$ and $\rho(t)<t$, respectively for any $t\in\mathbb{T}$.

Definition 2.2. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$ such that $\mu(t) = \sigma(t) - t$ is called graininess. When $\mathbb{T} = \mathbb{R}$, $\mu(t) \equiv 0$ and for $\mathbb{T} = \mathbb{Z}$, $\mu(t) \equiv 1$.

Definition 2.3. The mapping $\nu : \mathbb{T}_k \to \mathbb{R}_0^+$ such that $\nu(t) = t - \rho(t)$ is called backwards graininess.

Remark 2.1. :

1. The direction in a time scale has not been used in any symmetric manner (both in positive and negative directions), thus, we will consider the direction for a time scale $\mathbb{T}$ to be in the sense of increasing values of t for $t \in \mathbb{T}$.

2. If a time scale $\mathbb{T}$ has a maximal element, which is moreover left-scattered, then this point plays a particular role in several respects and therefore we call it degenerate. All other elements of $\mathbb{T}$ are called non-degenerate and the subset of non-degenerate points of $\mathbb{T}$ is denoted by $\mathbb{T}^k$. Since each closed subset of A of time scale $\mathbb{T}$ is also time scale, it is possible that A^k is formed.

Naturally $A^k = A$ is possible as long as A does not have a left-scattered maximum. Thus, $\mathbb{T}_k$ is defined as the set $\mathbb{T}_k = \mathbb{T}\ [\inf \mathbb{T}, \sigma(\inf \mathbb{T})]$ if $\inf \mathbb{T} < \infty$ and $\mathbb{T}_k = \mathbb{T}$ if $\inf \mathbb{T} = -\infty$ Likewise $\mathbb{T}^k$ is defined as the set

$$\mathbb{T}^k = \begin{cases} \mathbb{T}|[\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} \quad \text{if } \sup \mathbb{T} = \infty \end{cases}$$

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Finally, if $f : \mathbb{T} \to \mathbb{R}$, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f^{\sigma(t)}$ for all $t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$.

For all $t \in \mathbb{T}^k$, the following properties arise [5]:

(i) If f is delta differentiable at t, then f is continuous at t.
(ii) If f is left-continuous at t, and t is right scattered, then f is delta differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right dense, then f is delta differentiable at t, if and only if

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.

Similarly, for a nabla derivative of f at $t \in \mathbb{T}_k$, we have the following properties [13], [7]:

(a) If f is nabla differentiable at t, then f is continuous at t.
(b) If f is right continuous at t, and t is left-scattered, then f is nabla differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}$$

(c) If t is left-dense, then f is delta differentiable at t, if and only if

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(d) If f is nabla differentiable at t, then $f(\rho(t)) = f(t) - \nu(t) f^{\nabla}(t)$.

Definition 2.4. [7] A function $f : \mathbb{T} \to \mathbb{R}$ is called convex on $\mathbb{I}_{\mathbb{T}}$ if

$$f[\theta r + (1 - \theta)t] \leq \theta f(r) + (1 - \lambda) f(t) \tag{2.1}$$

for all $r, t \in \mathbb{I}_{\mathbb{T}}$ and $\theta \in [0, 1]$.

Definition 2.5. [10], [14]. A function f is called quasimonotonic if f is both quasiconcave and quasiconvex, i.e. for every $r, t \in \mathbb{R}^n$ and $\theta \in [0, 1]$

$$\min\{f(r), f(t)\} \leq f[\theta r + (1 - \theta)t] \leq \max\{f(r), f(t)\}. \tag{2.2}$$

Throughout this paper, $\mathbb{T}$ denotes a time scale and for any interval $I \in \mathbb{R}$ (closed or open), $\mathbb{I}_{\mathbb{T}}$=$I \cap \mathbb{T}$ is a time scale interval.

3 Results and Discussion

Results in this section are in the context of time scales. Unless otherwise stated the quasiconvex functions under this study assumes convexity and differentiability.

We begin by establishing some definitions on time scales.

Definition 3.1. A function $f : \mathbb{T} \to \mathbb{R}$ is called quasiconvex on $\mathbb{I}_{\mathbb{T}}$ if

$$f[\theta r + (1 - \theta)t] \leq \max\left[f(r), f(t)\right] \tag{3.1}$$

for all $r, t \in \mathbb{I}_{\mathbb{T}}$ and $\theta \in [0, 1]$.

Remark 3.1. From Definition 3.1, it is inferred that f is called quasiconvex if

$$f[\theta r + (1 - \theta)t] \leq f(r) \tag{3.2}$$

for $f(r) \geq f(t)$ at all convex combinations of t and r. Thus, f increases locally from its value at a point along the curve.

Definition 3.2. Let $f : \mathbb{T} \to \mathbb{R}$ on $\mathbb{I}_{\mathbb{T}}$ be such that the sublevel set $S_\alpha = \{t \in \mathbb{T} : f(t) \leq \alpha\}$ is convex. Then, $\bar{S_\alpha} = \{t \in \mathbb{T} : f(t) < \alpha\}$ holds in the strict case.

Let us also present the time scale version of Definition 2.5.

Lemma 3.1. *Let f be a monotonically increasing or decreasing quasiconvex function. Then*

$$f[\theta r + (1 - \theta)t] \leq \theta f(r) + (1 - \theta) f(t) \leq \max[f(r), f(t)] \tag{3.3}$$

holds for all $r, t \in \mathbb{I}_{\mathbb{T}}$ and $\theta \in [0, 1]$.

Proof. Let $r, t \in \mathbb{I}_{\mathbb{T}}$ and $r \geq t$, then $f(r) \geq f(t)$ for increasing function f. Thus for any $\theta \in [0,1]$, with the analysis on the vertical axis of the cartesian plane yields inequality (3.3). $\square$

Theorem 3.2. *Let $f : \mathbb{T} \to \mathbb{R}$ be an increasing convex function such that $t \leq s \leq r$ and $r \geq t$ with $s = \theta r + (1-\theta)t$. Then f is quasiconvex if*

$$(t-r)f(s) + (r-s)f(t) + (s-t)f(r) \geq 0$$

holds for all $s, t, r \in \mathbb{I}_{\mathbb{T}}$.

Proof. From $s = \theta r + (1-\theta)t$, we have

$$\theta = \frac{s-t}{r-t} \quad \textit{and} \quad 1-\theta = \frac{r-s}{r-t}. \tag{3.4}$$

Thus, by convexity, we have

$$f(s) = f[\theta r + (1-\theta)t] \leq \theta f[r] + (1-\theta)f[t] \tag{3.5}$$

Using Lemma 3.1, the inequality (3.5) becomes

$$f(s) \leq \theta f(r) + (1-\theta)f(t) \leq \max\{f(r), f(t)\} \tag{3.6}$$

Since f is increasing and $\max\{f(r), f(t)\} = f(r)$, then Substituting (3.4) into (3.6) yields

$$f(s) \leq \frac{s-t}{r-t} f(r) + \frac{r-s}{r-t} f(t) \tag{3.7}$$

Rearranging (3.7) yields the required result. $\square$

Lemma 3.3. *Suppose $f : \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$. Then f is called quasiconvex if and only if the sublevel set*

$$S_\alpha = \{t \in \mathbb{T} : f(t) \leq \alpha\}$$

is convex for any $\alpha \in \mathbb{R}$.

Proof. Let f be quasiconvex and suppose that $t \in \mathbb{I}_{\mathbb{T}}$ is isolated. Then there exists $t_1, t_2 \in S_\alpha \in \mathbb{I}_{\mathbb{T}}$.

such that $f(t_1) \leq \alpha$ and $f(t_2) \leq \alpha$.

$\mathbb{I}_{\mathbb{T}}$ is a time scale interval. Let $\theta \in [0,1]$ and $t = \theta t_1 + (1-\theta)t_2 \in S_\alpha \in \mathbb{I}_T$.

Then $f(t) = f(\theta t_1 + (1-\theta)t_2) \leq \max\{f(t_1), f(t_2)$.

Thus $f(t) \leq \max\{\alpha, \alpha\} = \alpha$.

Hence, $t \in S_\alpha$ and so S_α is convex.

Conversely, suppose that $S_\alpha(f)$ is convex. Then there exists $t_1, t_2 \in S_\alpha \in \mathbb{I}_{\mathbb{T}}$ such that

$\theta t_1 + (1-\theta)t_2 \in S_\alpha$ for any $\theta \in [0,1]$.

Then $f\{\theta t_1 + (1-\theta)t_2\} \leq \max\{f(t_1), f(t_2)\}$.

Thus, f is quasiconvex.

Also consider the case where t is dense and f is quasiconvex, then there exists $[s,t]$ and $[t,t_1]$ in $\mathbb{I}_{\mathbb{T}}$ such that $f(s) \leq \alpha$ and $f(t_1) \leq \alpha$. Let $t = \theta s + (1-\theta)t_1$.

Thus

$$f(t) = f(\theta s + (1-\theta)t_1) \leq \max\{f(s), f(t_1)\}$$

implies $f(t) \leq \alpha$ and $t \in S_\alpha$ is convex.

The converse for convexity of S_α clearly implies f is quasiconvex. Lastly, assume that f is quasiconvex and consider that t is left scattered and right dense or right scattered and left dense. Then there exist $\rho(t), \sigma(t) \in S_\alpha \in \mathbb{I}_\mathbb{T}$ such that $\rho(t) < \sigma(t)$ with $f(\rho(t)) \leq \alpha$ and $f(\sigma(t)) \leq \alpha$. Thus,

$f\{\theta\rho(t) + (1-\theta)\sigma(t)\} \leq \max\{f(\rho(t)), f(\sigma(t))\}$.
$f\{\theta\rho(t) + (1-\theta)\sigma(t)\} \leq f(\sigma(t))$
$f\{\theta\rho(t) + (1-\theta)\sigma(t)\} \leq \alpha$

Hence, $\sigma(t) \in S_\alpha$ and thus S_α is convex. Conversely, suppose that S_α is convex. Then, there exist $\rho(t), \sigma(t) \in S_\alpha \in \mathbb{I}_\mathbb{T}$

such that $\theta\rho(t) + (1-\theta)\sigma(t)\ S_\alpha$

for $\theta \in [0, 1]$.

Thus, $f\{\theta\rho(t) + (1-\theta)\sigma(t)\} \leq \max\{f(\rho(t)), f(\sigma(t))\}$.

This concludes the proof. □

Theorem 3.4. *Suppose $f : \mathbb{I}_\mathbb{T} \to \mathbb{R}$ is a delta differentiable function on $\mathbb{I}_\mathbb{T}$ and if f^Δ is quasi monotone on $\mathbb{I}_T$, then f^Δ is quasiconvex on $\mathbb{I}_T$.*

Proof. We begin by first establishing that the function f is delta differentiable.
Let $x < y < z \in \mathbb{I}_\mathbb{T}$. Then there exists points $x_1, x_2 \in [x, y]$ and $y_1, y_2 \in [y, z]$ such that

$$f^\Delta(x_1) \leq \frac{f(y) - f(x)}{y - x} \leq f^\Delta(y_1) \leq \frac{f(z) - f(y)}{z - y} \leq f^\Delta(y_2) \tag{3.8}$$

Now for $x < x_2 < y_1$, equation (3.8) becomes

$$\frac{f(y) - f(x)}{y - x} \leq f^\Delta(x_2) \leq f^\Delta(y_1) \leq \frac{f(z) - f(y)}{z - y}, \text{ for nondecreasing } f^\Delta. \tag{3.9}$$

If $x > x_2 > y$, then equation (3.9) becomes

$$\frac{f(y) - f(x)}{y - x} \geq f^\Delta(x_2) \geq f^\Delta(y_1) \geq \frac{f(z) - f(y)}{z - y}, \text{ for nondecreasing } f^\Delta. \tag{3.10}$$

Thus f^Δ exists.

Next is to establish that f^Δ is quasi monotone, that is to say the sub level sets $S_\alpha(f^\Delta)$ and $S_\beta(-f^\Delta)$ are convex by Lemma 3.3.

Now, let $f^\Delta : \mathbb{I}_\mathbb{T} \to \mathbb{R}$ be such that $S_\alpha(f^\Delta) = \{t \in \mathbb{I}_\mathbb{T} : f^\Delta(t) \leq \alpha, \forall \alpha \in \mathbb{R}\}$.

Thus for any $s, t, \in S_\alpha$, we have $\theta s + (1-\theta)t \in S_\alpha$. Hence f^Δ is quasi monotone and by Definition 2.5, f^Δ is quasiconvex. □

Theorem 3.5. *A function $f : \mathbb{I}_\mathbb{T} \to \mathbb{R}$ is quasiconvex on $\mathbb{I}_\mathbb{T} = \mathbb{I} \cap \mathbb{T}$ if and only if there exists a quasiconvex $\overline{f} : \mathbb{I} \to \mathbb{R}$ such that $\overline{f}(t) = f(t) \quad \forall t \in \mathbb{I}_\mathbb{T}$.*

Proof. For the sufficient part, since there exists a quasi convexfunction $\overline{f}$ on $\mathbb{I}$ such that $\overline{f}(t) = f(t)$, then

$$f(\theta s + (1-\theta)t) \leq \theta f(s) + (1-\theta)f(t) \leq \max\{f(s), f(t)\}$$

for all $t \in \mathbb{I}_\mathbb{T}, s, t \in \mathbb{I}$ and $\theta \in [0, 1]$.

When $t, s, (\theta + (1-\theta)t) \in \mathbb{T}$, then we get the inequality (3.1) which is the quasiconvexity on $\mathbb{I}_\mathbb{T}$.

Thus,

$$\overline{f}(t) = \begin{cases} f(t) & \text{if } t \in \mathbb{I}_{\mathbb{T}} \\ f(s) + \frac{f(\sigma(s))-f(s)}{\mu(s)}(t-s) & \text{if } t \in (s, \sigma(s)), s \in \mathbb{I}_{\mathbb{T}} \text{ and s is right scattered} \end{cases} \tag{3.11}$$

For any $x, y \in \mathbb{I}_{\mathbb{T}}$ and $\theta \in [0.1]$, we have

$$\overline{f}(\theta x + (1-\theta)y) \leq \theta\overline{f}(x) + (1-\theta)\overline{f}(y). \tag{3.12}$$

For $x, y \in \mathbb{I}_{\mathbb{T}}$ and $y > \sigma(x)$, the chord joining $(x, f(x))$ and $(y, f(y))$ is above all points $(z, f(z))$, with $z \in \mathbb{I}_{\mathbb{T}}$. If $x \in \mathbb{I}_{\mathbb{T}}$ and $y \in \mathbb{I}/\mathbb{T}$ with $y \leq \sigma(x)$, then $(y, f(y))$ is on the chord from $(x, f(x))$ to $(\sigma(x), f(\sigma(x))$ and so are all the points $(\theta x + (1-\theta)y)$. If $y > \sigma(x)$, then we can find $z \in \mathbb{I}_{\mathbb{T}}$ such that $x < z$ and $z < y < \sigma(z)$ such that

$$\frac{f(x)-f(z)}{x-z} \leq \frac{f(x)-f(\sigma(x))}{x-\sigma(x)} \tag{3.13}$$

while associating f on $[z, \sigma(z)]$ we have

$$\frac{f(x)-f(z)}{x-z} \leq \frac{f(x)-f(y)}{x-y} \leq \frac{f(x)-f(\sigma(z))}{x-\sigma(z)}. \tag{3.14}$$

Thus for $\theta \in [0,1], [\theta x + (1-\theta)y] \in [x, z]$ such that $\overline{f}(\theta x + (1-\theta)y = f(\theta x + (1-\theta)y)$,

where $t = \theta x + (1-\theta)y \in \mathbb{I}_{\mathbb{T}}$. This concludes the proof. □

Theorem 3.6. *A quasiconvex function is continuous at $t \in [\alpha, \vartheta]$ if and only if it is upper and lower semi continuous at $t \in (\alpha, \vartheta)$.*

Proof. If $\alpha \geq s < t < \mu \geq \theta$, then

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(\mu)-f(t)}{\mu-t}. \tag{3.15}$$

Let $t \in (\alpha, \vartheta)$ and consider $\alpha \leq t < t_n < s \leq \theta$, where (t_n) is such that $t_n \to t$ as $n \to \infty$ and $s \in (\alpha, \vartheta)$ is fixed. Then we have

$$t_n = \lambda_n t + (1-\lambda_n)s \tag{3.16}$$

where

$$\lambda_n = \frac{t_n - s}{t-s} \to 1, \text{ as } n \to \infty. \tag{3.17}$$

Taking the limit superior of both sides of (3.17), we consequently have

$$\lim_{n\to\infty} \sup f(t_n) = \lim_{n\to\infty} \sup f(\lambda_n t + (1-\lambda_n)s) \leq \lim_{n\to\infty} \sup(\lambda_n f(t) + (1-\lambda_n)f(s). \tag{3.18}$$

$$\text{Thus } f(t) \geq \lim_{n\to\infty} \sup f(t_n). \tag{3.19}$$

Hence f is upper semi continuous at $t \in (\alpha, \vartheta)$. Similarly, suppose (t_n) is such that $t_n \to t$ and $n \to \infty$ and $\alpha \leq t_n < t < z \leq \theta$, we have

$$t = \mu_n t_n + (1-\mu_n)z \tag{3.20}$$

where

$$\mu_n = \frac{t - s}{t_n - z} \tag{3.21}$$

$$\mu_n^{-1} = \frac{t_n - z}{t - z} \to 1, \text{ as } n \to \infty. \tag{3.22}$$

Thus

$$f(t) = f(\mu_n t_n + (1 - \mu_n)z) \leq \mu_n f(t_n) + (1 - \mu_n)f(z). \tag{3.23}$$

Rewriting (3.23) yields

$$(\mu_n^{-1} f(t) \leq f(t_n) + (1 - \mu_n)\mu_n^{-1} f(z). \tag{3.24}$$

Taking the limit interior of both sides. we have

$$\lim_{n\to\infty} \inf(\mu_n^{-1} f(t)) \leq \lim_{n\to\infty} \inf(f(t_n) + (1 - \mu_n)\mu_n^{-1} f(z)). \tag{3.25}$$

Thus

$$f(t) \leq \lim_{n\to\infty} \inf f(t_n). \tag{3.26}$$

Therefore f is lower semi continuous at $t \in (\alpha, \vartheta)$.

Conversely, suppose that f is upper semi continuous at $t_0 \in (\alpha, \vartheta)$.

Then for every $\epsilon > 0$, there exists a neighborhood $\mathcal{U}$ of t_0 such that

$$f(t) \leq f(t_0) + \epsilon, \tag{3.27}$$

for all $t \in \mathcal{U}$ when $f(t_0) > -\infty$ and $f(t)$ tends to $-\infty$ as t tends to when $f(t_0) = -\infty$. This implies that

$$|f(t) - f(t_0)| < \epsilon, \text{ when } |t - t_0| < \delta, \tag{3.28}$$

where δ is a very small number. Thus f is continuous.

Similarly, if f is lower semi continuous, then for every $\epsilon > 0$,there exists $\mathcal{U}$ of t_0 such that

$$f(t) \geq f(t_0), \tag{3.29}$$

for all $t \in \mathbf{U}$ when $f(t_0) < +\infty$ and $f(t)$ tends to $+\infty$ as t tends to when $f(t_0) = +\infty$. Thus

$$\epsilon \geq f(t_0) - f(t). \tag{3.30}$$

Rewriting (3.30), we have

$$|f(t_0) - f(t)| \geq \epsilon \implies |-(f(t) - f(t_0))| \leq \epsilon. \tag{3.31}$$

Therefore,

$$|f(t) - f(t_0)| < \epsilon, \text{ when } |t - t_0| < \delta. \tag{3.32}$$

Hence f is continuous. □

Theorem 3.7. *Let $f : [q, r]_{\mathbb{T}} \to \mathbb{R}$ be quasiconvex function. Then for all $\alpha, \vartheta \in [q, r]_{\mathbb{T}}$ with $\alpha < \vartheta$, we have $f_-^{\nabla}(\alpha) \leq f_+^{\Delta}(\alpha) \leq f_-^{\nabla}(\vartheta) \leq f_+^{\Delta}(\vartheta)$ and hence both f_-^{∇} and f_+^{Δ} exists and they are increasing on $[q, r]_{\mathbb{T}}$.*

Proof. Let $x < y < \alpha < z \in [q, r]_{\mathbb{T}}$. Then,

$$\frac{f(x) - f(\alpha)}{x - \alpha} \leq \frac{f(y) - f(\alpha)}{y - \alpha} < \frac{f(z) - f(\alpha)}{z - \alpha}. \tag{3.33}$$

Suppose α is right scattered and left dense and a y approaches α, we have

$$\lim_{y\to\alpha} \frac{f(y) - f(\alpha)}{y - \alpha} = f^{\nabla}(\alpha) \leq \frac{f(z) - f(\alpha)}{z - \alpha}. \tag{3.34}$$

We notice that,

$$\mathcal{F} : [q, r]_{\mathbb{T}} \to \mathbb{R}, \mathcal{F}(\alpha) = \frac{f(y) - f(\alpha)}{y - \alpha}, \tag{3.35}$$

is nondecreasing and bounded above as a function of α. If we substitute $z = \sigma(\alpha)$ into (3.35), we get

$$f^{\nabla}(\alpha) \leq \frac{f(\sigma(\alpha)) - f(\alpha)}{\sigma(\alpha) - \alpha} = f^{\Delta}(\alpha). \tag{3.36}$$

$$f^{\nabla}(\alpha) \leq f^{\Delta}(\alpha). \tag{3.37}$$

A similar argument will give the same conclusion for α being left scattered and right dense. Assume that α is left scattered and right dense and z approaches α, we have

$$\lim_{z \to \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} = f^{\Delta}(\alpha) \geq \frac{f(y) - f(\alpha)}{y - \alpha}. \tag{3.38}$$

If we put $y = \rho(\alpha)$ into (3.38), we arrive at

$$f^{\Delta}(\alpha) \geq \frac{f(\rho(\alpha)) - f(\alpha)}{\rho(\alpha) - \alpha} = f^{\nabla}(\alpha). \tag{3.39}$$

Re-writing (3.38), we get (3.37). If α is an isolated point, that is, $\rho(\alpha) < \alpha < \sigma(\alpha)$, then

$$\frac{f(\rho(\alpha)) - f(\alpha)}{\rho(\alpha) - \alpha} \leq \frac{f(\sigma(\alpha)) - f(\alpha)}{\sigma(\alpha) - \alpha} \equiv f^{\nabla}(\alpha) \leq f^{\Delta}(\alpha). \tag{3.40}$$

If α is dense, that is, $\rho(\alpha) = \alpha = \sigma(\alpha)$, then making y and z to tend to α and using the nondecreasing function f, we get

$$\lim_{y \to \alpha, y < \alpha} \frac{f(y) - f(\alpha)}{y - \alpha} \leq \lim_{z \to \alpha, z > \alpha} \frac{f(z) - f(\alpha)}{z - \alpha}. \tag{3.41}$$

Thus, for every $\alpha \in [q, r]_{\mathbb{T}}$, we have

$$f_{-}^{\nabla}(\alpha) \leq f_{+}^{\Delta}(\alpha). \tag{3.42}$$

Conversely, for $\alpha < z < w < \vartheta$, we have

$$\frac{f(z) - f(\alpha)}{z - \alpha} \leq \frac{f(w) - f(\alpha)}{w - \alpha} \leq \frac{f(w) - f(\vartheta)}{w - \vartheta}. \tag{3.43}$$

Concise form of (3.43) gives

$$f_{+}^{\Delta}(\alpha) \leq f_{-}^{\nabla}(\vartheta). \tag{3.44}$$

Combining (3.43) and 3.44) concludes the proof. □

4 Conclusions

The study established that the structure underlying quasiconvex functions can be represented in the context of time scales. The study further established the existence of some properties such as set relations, semicontinuity, differentiability and inequalities of quasiconvex functions in the domain of time scales.

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Competing Interests

Authors have declared that no competing interests exist.

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