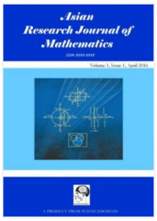

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On Series of Transformed Zeta Function

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

We transform a zeta function to the alternative sum as $\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$ and represent it as some series, for example

$$\sum_{n=1}^{\infty} \frac{(s)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}}, \qquad \sum_{n=1}^{\infty} \frac{(s+1)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}},$$

etc., where $(s)_n = s(s+1)\cdots(s+n-1)$, $Re(s) > 1$, and we obtain their formulas.

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1 Introduction

For $Re(s) > 1$ the Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.1}$$

It is well known that $\zeta(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$ with residue 1. Moreover, $\zeta(0) = -1/2$. [1] gives an elementary proof of the classical result

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In [2] Ewell modifies Boo Rim Choe's method to show a new series representation of $\zeta(3)$, namely,

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.$$

In this paper we set an alternative sum as

$$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \tag{1.2}$$

and

$$\xi(s) = \sum_{n=1}^{\infty} \left(\frac{-4}{n}\right) \frac{1}{n^s} \tag{1.3}$$

where the Legendre-Jacobi-Kronecker symbol for discriminant -4, that is for $n \in \mathbb{N}$

$$\left(\frac{-4}{n}\right) := \begin{cases} 0, & \text{if} \quad n \equiv 0 \pmod 2, \\ 1, & \text{if} \quad n \equiv 1 \pmod 4, \\ -1, & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

Then we obtain

Theorem 1.1. *We have*

(a)

$$\sum_{n=1}^{\infty} \frac{(s+1)_{2n-1}\zeta^*(s+2n)}{(2n-1)!2^{2n}} = -2^s\xi(s+1) + 2^{s-1}, \qquad Re(s) > 0,$$

(b)

$$\sum_{n=1}^{\infty} \frac{(s)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}} = -2^{s-1} - \zeta^*(s), \qquad Re(s) > 1,$$

(c)

$$\sum_{n=1}^{\infty} \frac{(s+1)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}} = -\zeta^*(s) - 2^s\xi(s+1), \qquad Re(s) > 1,$$

where $(s)_n = s(s+1)\cdots(s+n-1)$ *and* $(s)_0 = 1$.

Theorem 1.2. *We have*

$$\sum_{n=1}^{\infty} \frac{\zeta^*(2n+1)}{2^{2n}} = -1 + \ln 2.$$

2 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. For $0 < a \leq 1$ and $Re(s) > 1$ the function $\zeta^*(s,a)$ is defined by

$$\zeta^*(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s}.$$

In fact, $\zeta^*(s,a)$ is similar to the Hurwitz zeta function, named after Adolf Hurwitz, which is defined for complex arguments s with $Re(s) > 1$ and q with $Re(q) > 0$ by

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}.$$

Now we set

$$\mu(s,a) = \zeta^*(s,a) - \zeta^*(s,1-a), \qquad 0 < a < 1, \quad Re(s) > 1.$$

Then we have

$$\begin{aligned}
\mu(s,a) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+a)^s} - \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1-a)^s} \\
&= \frac{1}{a^s} + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+a)^s} + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-a)^s} \\
&= \frac{1}{a^s} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s} \left(1 + \frac{a}{m}\right)^{-s} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s} \left(1 - \frac{a}{m}\right)^{-s}.
\end{aligned}$$

Since

$$\begin{aligned}
\left(1 + \frac{a}{m}\right)^{-s} &= \sum_{n=0}^{\infty} \frac{(-1)^n (s)_n}{n!} \left(\frac{a}{m}\right)^n, \\
\left(1 - \frac{a}{m}\right)^{-s} &= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(\frac{a}{m}\right)^n,
\end{aligned}$$

thus the above identity can be written as

$$\begin{aligned}
\mu(s,a) &= \frac{1}{a^s} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m^s} \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left(\frac{a}{m}\right)^n ((-1)^n + 1) \\
&= \frac{1}{a^s} + 2\sum_{n=0}^{\infty} \frac{(s)_{2n} a^{2n}}{(2n)!} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^{s+2n}} \qquad (2.1) \\
&= \frac{1}{a^s} + 2\sum_{n=0}^{\infty} \frac{(s)_{2n} \zeta^*(s+2n)}{(2n)!} a^{2n}.
\end{aligned}$$

Similarly with

$$\lambda(s,a)=\zeta^*(s,a)+\zeta^*(s,1-a), \qquad 0<a<1, \quad Re(s)>1.$$

we obtain

$$\lambda(s,a)=\frac{1}{a^s}-2\sum_{n=1}^{\infty}\frac{(s)_{2n-1}\zeta^*(s+2n-1)}{(2n-1)!}a^{2n-1}. \tag{2.2}$$

(a) Letting $a=\frac{1}{2}$ and changing s into $s+1$ in (2.2), we obtain

$$\begin{aligned}2^{s+1}-4\sum_{n=1}^{\infty}\frac{(s+1)_{2n-1}\zeta^*(s+2n)}{(2n-1)!2^{2n}} &= \lambda(s+1,\frac{1}{2})\\ &= 2\zeta^*(s+1,\frac{1}{2})\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\frac{1}{2})^{s+1}}\\ &= 2^{s+2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{s+1}}\\ &= 2^{s+2}\xi(s+1).\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{(s+1)_{2n-1}\zeta^*(s+2n)}{(2n-1)!2^{2n}}=-2^s\xi(s+1)+2^{s-1}.$$

(b) Letting $a=\frac{1}{2}$ in (2.1), we have

$$2^s+2\sum_{n=0}^{\infty}\frac{(s)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}}=\mu(s,\frac{1}{2})=0$$

and so

$$\sum_{n=1}^{\infty}\frac{(s)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}}=-2^{s-1}-\zeta^*(s).$$

(c) Adding Theorem 1.1 (a) to (b) and noticing

$$(s)_{2n}+2n(s+1)_{2n-1}=(s+1)_{2n}$$

we deduce that

$$\sum_{n=1}^{\infty}\frac{(s+1)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}}=-\zeta^*(s)-2^s\xi(s+1).$$

□

Lemma 2.1. *We have*

(a)

$$\xi(2) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{n\zeta^*(2n+1)}{2^{2n}},$$

(b)

$$\sum_{n=1}^{\infty} \frac{(2n-1)\zeta^*(2n)}{2^{2n}} = -\frac{1}{2}.$$

Proof. (a) We take $s = 1$ in Theorem 1.1 (a) then we obtain

$$\begin{aligned} -2\xi(2) + 1 &= \sum_{n=1}^{\infty} \frac{(2)_{2n-1}\zeta^*(1+2n)}{(2n-1)!2^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdots (2n)\zeta^*(2n+1)}{(2n-1)!2^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{2n\zeta^*(2n+1)}{2^{2n}} \end{aligned}$$

and so

$$\xi(2) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{n\zeta^*(2n+1)}{2^{2n}}.$$

(b) Similarly, we replace s with 2 in Theorem 1.1 (b) :

$$\begin{aligned} -2 &= \zeta^*(2) + \sum_{n=1}^{\infty} \frac{(2)_{2n}\zeta^*(2+2n)}{(2n)!2^{2n}} \\ &= \zeta^*(2) + \sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdots (2n+1)\zeta^*(2n+2)}{(2n)!2^{2n}} \\ &= \zeta^*(2) + \sum_{n=1}^{\infty} \frac{(2n+1)\zeta^*(2n+2)}{2^{2n}} \\ &= \sum_{n=0}^{\infty} \frac{(2n+1)\zeta^*(2n+2)}{2^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{(2n-1)\zeta^*(2n)}{2^{2n-2}} \end{aligned}$$

and so

$$\sum_{n=1}^{\infty} \frac{(2n-1)\zeta^*(2n)}{2^{2n}} = -\frac{1}{2}.$$

□

From (1.1) and (1.2) we note that

$$\begin{aligned}\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} &= \sum_{n=1}^{\infty} \frac{1}{(2n)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \\ &= 2^{-s} \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \\ &= 2^{-s}\zeta(s) - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}\end{aligned}$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = 2^{-s}\zeta(s) - \zeta^*(s).$$

Here we yield that

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} &= \sum_{n=1}^{\infty} \frac{1}{(2n)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \\ &= 2^{-s}\zeta(s) + 2^{-s}\zeta(s) - \zeta^*(s) \\ &= 2^{1-s}\zeta(s) - \zeta^*(s),\end{aligned}$$

which leads that

$$\zeta^*(s) = (2^{1-s} - 1)\zeta(s). \tag{2.3}$$

Proposition 2.1. *(See [3], [4])*

(a)

$$\pi^{1-s}\zeta(s) = 2^s\Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2},$$

(b)

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin \pi s}, \qquad s \notin \mathbb{Z}.$$

Proof of Theorem 1.2. Letting $s \to 1$ in Theorem 1.1 (b) and recalling Eq. (2.3) and Proposition 2.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\zeta^*(2n+1)}{2^{2n}} &= \lim_{s\to 1} \sum_{n=1}^{\infty} \frac{s(s+1)\cdots(s+2n-1)\zeta^*(s+2n)}{(2n)!2^{2n}} \\ &= \lim_{s\to 1} \sum_{n=1}^{\infty} \frac{(s)_{2n}\zeta^*(s+2n)}{(2n)!2^{2n}} \\ &= \lim_{s\to 1} \left\{-2^{s-1} - \zeta^*(s)\right\} \\ &= -1 - \lim_{s\to 1} (2^{1-s} - 1)\zeta(s) \\ &= -1 - \lim_{s\to 1} (2^{1-s} - 1) \cdot 2^s \pi^{s-1} \Gamma(1-s)\zeta(1-s)\sin\frac{\pi s}{2} \\ &= -1 - \lim_{s\to 1} (2^{1-s} - 1) \cdot 2^s \pi^{s-1} \frac{\pi}{\Gamma(s)\sin \pi s} \zeta(1-s)\sin\frac{\pi s}{2}\end{aligned}$$

$$
\begin{aligned}
&= -1 - \lim_{s\to 1} 2^s \pi^{s-1} \frac{\pi}{\Gamma(s)} \zeta(1-s) \sin\frac{\pi s}{2} \cdot \lim_{s\to 1} \frac{2^{1-s}-1}{\sin \pi s} \\
&= -1 + \pi \left(- \lim_{s\to 1} \frac{2^{1-s}\ln 2}{\pi \cos \pi s} \right) \\
&= -1 + \ln 2.
\end{aligned}
$$

□

3 Conclusion

In this article we modify the Riemann zeta function and consider their infinite sums.

Competing Interests

Author has declared that no competing interests exist.

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