Asian Research Journal of Mathematics

8(4): 1-11, 2018; Article no.ARJOM.39819

ISSN: 2456-477X

Fixed Point Theorems in Partial b-Metric Spaces Using Contractive Conditions

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Authors' contributions

This work was carried out in collaboration between all authors. Author PK designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors ZKA and AG managed the analyses of the study. Author AG managed the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/39819
Editor(s):
(1) Ozgur Ege, Department of Mathematics, Faculty of Science & Letters, Celal Bayar University, Manisa, Turkey.
Reviewers:
(1) Krisnandari Ekowati, Nusa Cendana University, Indonesia.
(2) Nihal Taş, Balikesir University, Turkey.
(3) W. Obeng-Denteh, KNUST, Ghana.
(4) Anita Tomar, V. S. K. C. Government P. G. College, India.
Complete Peer review History: http://www.sciencedomain.org/review-history/23656

Received: 2nd January 2018
Accepted: 6th March 2018
Published: 15th March 2018

Original Research Article

Abstract

In this paper, we prove a fixed point theorem in the framework of complete partial b-metric space. Inspiring from Suzuki and Piri-Afshari. The article also includes an example which shows the validity of our result.

Keywords: Fixed point; b-metric space; partial metric space; partial b- metric space and contraction.

2010 AMS classification: 54H25, 47H10.

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1 Introduction

Fixed point theory is one of the most important tools in the development of mathematics because it plays an essential role in applications of many branches of mathematics. For this reason, several researchers studied the fixed point of contractive maps (see for example [1] and references therein).

In 1992, Polish mathematician Banach [2] proved a very important result regarding a contraction mapping, known as Banach contraction principle. It is one of the fundamental results in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations of metric spaces and Banach contraction principle; see [3-12] and references therein.

In this sequel, in 1989, Bakhtin [13] introduced the concept of b-metric spaces and presented the contraction mapping in b-metric spaces that is generalization of the Banach contraction principle in metric spaces (see also Czerwik [14]. After that, fixed point results in b-metric spaces were studied by several researchers; see [15-25] and references therein. On the other hand, Matthews [26,27] introduced the notion of a partial metric space which is a generalization of usual metric space. Also, he generalized the Banach contraction principle in the context of complete partial metric spaces. Recently, many researchers have focused on partial metric spaces and obtained many useful fixed point results in these spaces (see [28-30,16,31,32,21]) and references therein). Very recently, Shukla [33] introduced partial b-metric spaces as a generalization of both bmetric spaces and partial metric spaces. Moreover, he proved Banach contraction principle as well as the Kannan type fixed point theorem in partial b-metric spaces.

Several authors have obtained many interesting extensions and generalizations of metric spaces (see [34-38]) and references therein.

In this paper, motivated and inspired by ideas of some recent papers such as Mustafa et al. [39], Suzuki [4], Wong [40] and Shukla [33], we obtain some fixed point theorems in partial b-metric spaces. Our result is the generalization of the result announced by Suzuki [4], Wong [41] and Shukla [33,41] Piri.H, Afshari and some others. Throughout this paper, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers and the set of positive integers, respectively. We proved a fixed point theorem in partial b-metric space.

Before proving our main results, we define some definitions, basic properties, examples and Lemmas of b-metric space, partial metric space and partial b-metric space needed in the sequel.

Definition 1.1 [26]: Let X be a nonempty set and s ≥ 1 be a given real number. a mapping $d: X \times X \to \mathbb{R}^+$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(b_1) $d\,(x, y) = 0$ *if and only if* $x = y$;

(b_2) $d\,(x, y) = d(y, x)$;

(b_3) $d\,(x, y) \leq s[d(x, z) + d(z, y)]$.

In this case, the pair (X,d) is called a b- metric space (with constant s).

Remark: The class of b-metric space is the generalization of metric space. If $s = 1$, then b-metric space converts into metric space.

Example1.2: Let (X, d) be a metric space and $\sigma(x, y) = (d(x, y))^p$ *where* $p > 1$, Then σ is a b-metric for $s = 2^{p-1}$.

Proof: (i) For $x = y \, then \, \sigma(x,x) = (d(x,x))^p = 0$

$(ii)\sigma(x,y) = (d(x,y))^p = (d(y,x))^p = \sigma(y,x)$

(iii) if $p > 1$ then by then convexity of the function $f(x) = x^p \Rightarrow (\frac{a+b}{2})^p \leq \frac{1}{2}(a^p + b^p) \Rightarrow (a+b)^p = 2^{p-1}(a^p + b^p)$. So for all $x, y, z \in X$, we have

$$\sigma(x,y) = (d(x,y))^p \leq (d(x,z) + d(z,y))^p \leq 2^{p-1}(d(x,z))^p + (d(z,y))^p \leq 2^{p-1}\{\sigma(x,z) + \sigma(z,y)\}$$

Hence (X, σ) is a b-metric space.

Example 1.3: Let $X = \mathbb{R}$ be the set of real number and $d(x,y) = |x - y|$ a usual metric. Then $\sigma(x,y) =$

$|x - y|$ is a b-metric space for $k = 2$ but not for $\mathbb{R}$.

Example 1.4: Let $X = [0, \infty)$ and $d: X \times X \to \mathbb{R}^+$ is a mapping defined as $d(x,y) = |x - y|^p$.Then(X, d) is a

b-metric space. Where p is a real number such that p > 1.

Proof: (a) $d(x,y) = 0 \Rightarrow |x - y|^p = 0 \Rightarrow x = y$

(b) $d(x,y) = |x - y|^p = |y - x|^p = d(y,x)$

(c) Let $u = x - z, v = z - y$, then $d(x,y) = |x - y|^p = |u + v|^p \leq |u|^p + |v|^p \leq (2 \, max|u|, |v|)^p$

$$\leq 2^p(|u|^p + |v|^p) = 2^p(|x - z|^p + |z - y|^p)$$

$\Rightarrow (X, d)$ is a b-metric space with a constant $s = 2^p$.

Definition 1.5 [26]: Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a parietal metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

$(\boldsymbol{p_1})$ $x = y \; if \; and \; only \; if \; p(x,x) = p(x,y) = p(y,y)$;
$(\boldsymbol{p_2})$ $p(x,x) \leq p(x,y)$;
$(\boldsymbol{p_3})$ $p(x,y) = p(x,y)$;
$(\boldsymbol{p_4})$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

In this case, the pair (X,d) is called a partial metric space.

Example 1.6: Let $X = \{ [a,b], a, b \in R, a \leq b\}$ and define as $p([a,b],[c,d]) = \max(b,d) - \min(a,c)$, then
(X, p) is a partial metric spaces.

Definition 1.7. [33]: Let X be nonempty set and s ≥ 1 be a given real number, A mapping $p_b: X \times X \to \mathbb{R}^+$ is said to be a partial b-metric on X if for $x, y, z \in X$ the following conditions are satisfied:

(p_{b^1}) $x = y \; if \; and \; only \; if \; p_b(x,x) = p_b(x,y) = p_b(y,y)$;
(p_{b^2}) $p_b(x,x) \leq p_b(x,y)$;
(p_{b^3}) $p_b(x,y) = p_b(x,y)$;
(p_{b^4}) $\boldsymbol{p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)}$.

In this case, the pair (X, $\boldsymbol{p_b}$) **is called** a partial b-metric space (with constant s).

Remark: Every b-metric space is a partial b-metric space for $s = 1$ and similarly every partial metric space is a partial b-metric space for $s = 1$, but the converse is not true.

Example 1.8: Here is an example that shows a partial b- metric space is not partial metric space.

Let $X = \{1,2,3\}$ *and* $p_b(1,1) = 1$, $p_b(1,2) = p_b(2,1) = 6$, $p_b(2,2) = 5, p_b(3,2) = p_b(2,3) = 4$,

$p_b(1,3) = p_b(3,1) = 9, p_b(3,3) = 9$.

$p_b(1,1) = p_b(2,2) = p_b(3,3) \neq 0$

$p_b(3,2) + p_b(2,1) - p_b(2,2) = 6 + 4 - 5 = 5 < 9 = p_b(3,1) \Rightarrow (X,p)$ is not a partial metric space.

$2[p_b(3,2) + p_b(2,1)] - p_b(2,2) = 20 - 5 = 15 > 9 = p_b(3,1) \Rightarrow (X,p)$ is a partial b- metric space.

Definition 1.9 [33]: Let $(X,\ p_b)$ be a Partial b- metric space with constant s**, and** let $\{x_n\}$ be a sequence in X and $x \in X$, then :

(a) The sequence $\{x_n\}$ is said to be converge to $x \in X\ if\ p_b(x,x) = \lim_{n\to\infty} p_b(x,x_n)$;

(b) The sequence $\{x_n\}$ is said to be Cauchy in (X,p_b) , if $\lim_{n,m\to\infty} p_b(x_n,x_m)$ exists and is finite;

(c) $(X,\ p_b)$ is said to be complete if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n\to\infty} p_b(x_n\ x) = 0 \Leftrightarrow p_b(x,x) = \lim_{n,\to\infty} p_b(x_n,x) = \lim_{n,m\to\infty} p_b(x_n,x_m)$$

2 Main Result

Theorem 2.1: Let $(X,\ p_b)$ be a complete partial b- metric space with constant $s \geq 1$ and $T: X \to X$ be a self map on X. Suppose $M \in [0,\infty)$ and functions $\alpha_i \in [0,\infty) \to [0,\infty)$, *where* $i = 1,2,3,4$, such that

(2.2) Functions α_i are upper semi continuous from right.
(2.3) $\alpha_1 + 2\,\alpha_2 + 2\,\alpha_3 + M\alpha_4 < \frac{1}{2s^2}t\ \ \forall\, t > 0$
(2.4) $\frac{1}{2s}p_b(x,Tx) < p_b(x,y) \Rightarrow$

$$p_b(x,y)\,p_b(Tx,Ty) \leq \alpha_1(p_b(x,y))^2 + \alpha_2[p_b(x,y)p_b(x,x) + p_b(x,y)p_b(x,Tx)]$$
$$+\alpha_3[p_b(x,y)p_b(y,y) + p_b(x,y)p_b(y,Ty)] + \alpha_4 Max\{p_b(x,y), p_b(x,x), p_b(y,y)\}p_b(x,y)$$

for all $x,y \in X$. Then T has a unique fixed point in X.

Proof: First we shall prove that if T has a fixed point, then it is unique. Let $x,y \in X$ be two distinct points of T such that $x \neq y \Rightarrow Tx \neq Ty$. Therefore $p_b(x,y) > 0$. If $p_b(x,x) = 0$, then we have $0 = \frac{1}{2s}p_b(x,x) = \frac{1}{2s}p_b(x,Tx) < p_b(x,y)$. If $p_b(x,x) > 0$. From p_2 we conclude that $\frac{1}{2s}p_b(x,Tx) = \frac{1}{2s}p_b(x,x) < p_b(x,x) < p_b(x,y)$. Using the inequality (2.2), (2.3) and (2.4), we have

$$\begin{aligned}(p_b(x,y))^2 &= p_b(x,y)\,p_b(Tx,Ty) \leq \alpha_1(p_b(x,y))^2 + \alpha_2[p_b(x,y)p_b(x,x) + p_b(x,y)p_b(x,Tx)]\\ &\quad +\alpha_3[p_b(x,y)p_b(y,y) + p_b(x,y)p_b(y,Ty)] + \alpha_4 Max\,\{p_b(x,y), p_b(x,x), p_b(y,y)\}p_b(x,y)\\ &= \alpha_1(p_b(x,y))^2 + \alpha_2[p_b(x,y)p_b(x,x) + p_b(x,y)p_b(x,x)]\\ &\quad +\alpha_3[p_b(x,y)p_b(y,y) + p_b(x,y)p_b(y,y)] + M\alpha_4\{p_b(x,y)\}p_b(x,y)\end{aligned}$$

$$\leq \alpha_1(p_b(x,y))^2 + \alpha_2[p_b(x,y)p_b(x,y) + p_b(x,y)p_b(x,y)]$$
$$+\alpha_3[p_b(x,y)p_b(x,y) + p_b(x,y)p_b(x,y)] + M\alpha_4\{p_b(x,y)\}p_b(x,y)$$

$$(p_b(x,y))^2 \leq [\alpha_1 + 2\alpha_2 + 2\alpha_3 + M\alpha_4](p_b(x,y))^2$$

$$< \tfrac{1}{2s^2}p_b(x,y) < p_b(x,y)$$

This is a contradiction. So $x = y$. Choosing $x_0 \in X$ such that

$$x_1 = Tx_0, x_2 = Tx_1\ , x_3 = Tx_2, \ldots . x_{n+1} = Tx_n\ ; \quad P_n = p_b(\ x_n, Tx_n\ )$$

If there exits an $n \in N$, such that $P_n = 0$, then proof is finished. So we assume that $for\ all\ \ n \in N,\ \ P_n = p_b(x_n, Tx_n) = p_b(x_n, x_{n+1}) > 0 \Rightarrow \frac{1}{2s}p_b(x_n, Tx_n) < p_b(x_n, Tx_n)$.

$$p_b(x_n,Tx_n)p_b(Tx_n,T^2x_n) \leq \alpha_1\big(p_b(x_n,Tx_n)\big)^2 + \alpha_2 p_b(x_n,Tx_n)[p_b(x_n,x_n) + p_b(x_n,Tx_n)]$$

$$+\alpha_3\ p_b(x_n,\ Tx_n)[p_b(Tx_n,\ Tx_n) + p_b(Tx_n,\ T^2x_n)]$$

$$+\alpha_4 Max\{p_b(x_n,Tx_n), p_b(x_n,x_n), (Tx_n,Tx_n)\}p_b(x_n,Tx_n)$$
$$\leq \alpha_1\big(p_b(x_n,Tx_n)\big)^2 + \alpha_2 p_b(x_n,Tx_n)[\ 2s\ p_b(x_n,Tx_n) + p_b(x_n,Tx_n)]$$

$$+\alpha_3\ p_b(x_n,Tx_n)[\ 2s\ p_b(Tx_n,T^2x_n) + p_b(Tx_n,T^2x_n)] + \alpha_4 M\{p_b(x_n,Tx_n)\}p_b(x_n,Tx_n)$$
$$p_b(x_n,T_n)p_b(xTx_n,T^2x_n) \leq \alpha_1\big(p_b(x_n,Tx_n)\big)^2 + (2s+1)\alpha_2\ (p_b(x_n,Tx_n))^2$$
$$+(2s+1)\alpha_3\ p_b(x_n,Tx_n)p_b(Tx_n,T^2x_n) + \alpha_4 M\{p_b(x_n,Tx_n)\}p_b(x_n,Tx_n)$$

$$(1-(2s+1)\alpha_3)\ p_b(x_n,Tx_n)p_b(Tx_n,T^2x_n) \leq (\alpha_1 + (2s+1)\alpha_2 + \alpha_4 M)\ (p_b(x_n,Tx_n))^2$$

$$(1-(2s+1)\alpha_3)\ P_n P_{n+1} \leq (\alpha_1 + (2s+1)\alpha_2 + \alpha_4 M)\ {P_n}^2$$

$$P_{n+1} \leq \frac{(\alpha_1 + (2s+1)\alpha_2 + \alpha_4 M)}{(1-(2s+1)\alpha_3)}\ P_n$$

Since $\alpha_1 + (2s+1)\alpha_2 + (2s+1)\alpha_3 + \alpha_4 M < 2s\,[\alpha_1 + 2\alpha_2 + 2\,\alpha_3 + \alpha_4 M] < 2s\frac{1}{2s^2}t < \frac{1}{s}t \ \ for\ all\ \ t > 0$ $\Rightarrow \frac{(\alpha_1+(2s+1)\alpha_2+\alpha_4 M)}{(1-(2s+1)\alpha_3)} < 1$, therefore $p_b(Tx_n, T^2x_n) = P_{n+1} \leq\ P_n = p_b(x_n, Tx_n)\ \ for\ all\ \ n \in N$ implies that $\{P_n\}$ is a decreasing sequence which is bounded from below and $n \in [0,1)$.Hence $\{P_n\}$ converses to a point $p \in [0,1)$. So we have

$$p = \lim_{n\to\infty} P_n \leq \limsup_{n\to\infty} P_n \leq \limsup_{n\to\infty} \frac{(\alpha_1(P_n) + (2s+1)\alpha_2(P_n) + M\alpha_4(P_n))}{(1-(2s+1)\alpha_3(P_n))}\ P_n$$
$$\leq \frac{(\alpha_1(p) + (2s+1)\alpha_2(p) + M\alpha_4(p))}{(1-(2s+1)\alpha_3(p))}\ p < p$$

This is a contradiction. Therefore

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} p_b(x_n, Tx_n) = 0$$

Now we shall show that $\mathbf{\lim_{n,m\to\infty} p_b(x_n, x_m) = 0}$.

We shall prove it via contradiction. Assuming$\exists\ \varepsilon > 0$, there exist two sequences $p(n), q(n) \in [01)$ such that $p(n) > q(n) > n,\ p_b\big(x_{p(n)},\ x_{q(n)}\big) \geq \varepsilon\ and\ p_b\big(x_{p(n)-1},\ x_{q(n)}\big) < \varepsilon\ for\ all\ \ n \in N$. So we have
$\varepsilon \leq p_b\big(x_{p(n)}, x_{q(n)}\big) \leq s\big[p_b\big(x_{p(n)}, x_{p(n)-1}\big) + p_b\big(x_{p(n)-1}, x_{q(n)}\big)\big] - p_b\big(x_{p(n)-1}, x_{p(n)-1}\big)$

$$\leq s[p_b(x_{p(n)}, x_{p(n)-1}) + p_b(x_{p(n)-1}, x_{q(n)})] \leq sp_b(x_{p(n)}, x_{p(n)-1}) + s\varepsilon$$

Also we have

$$\varepsilon \leq \liminf_{n\to\infty} p_b(x_{p(n)}, x_{q(n)}) \leq \limsup_{n\to\infty} p_b(x_{p(n)}, x_{q(n)}) \leq s\varepsilon$$

As we have already prove that $\lim_{n\to\infty} p_b(x_n, Tx_n) = 0 \Rightarrow \lim_{n\to\infty} p_b(x_{p(n)}, Tx_{p(n)}) = 0$

$$\Rightarrow \frac{1}{2s} p_b(x_{p(n)}, Tx_{p(n)}) < \varepsilon$$

Again we have

$$\frac{1}{2s} p_b(x_{p(n)}, Tx_{p(n)}) < p_b(x_{p(n)+1}, x_{q(n)+1}) \leq p_b(Tx_{p(n)}, Tx_{q(n)})$$

Using the inequality (2.4) we have

$$p_b(x_{p(n)}, x_{q(n)}) p_b(Tx_{p(n)}, Tx_{q(n)})$$

$$\leq \alpha_1 \left(p_b(x_{p(n)}, x_{q(n)})\right)^2 + \alpha_2 p_b(x_{p(n)}, x_{q(n)})[p_b(x_{p(n)}, x_{p(n)}) + p_b(x_{p(n)}, Tx_{p(n)})]$$

$$+\alpha_3 p_b(x_{n(p)}, Tx_{q(n)})[p_b(x_{q(n)}, x_{q(n)}) + p_b(x_{q(n)}, Tx_{q(n)})]$$

$$+M\alpha_4\{p_b(x_{p(n)}, x_{q(n)}) p_b(x_{p(n)}, x_{p(n)}) p_b(x_{q(n)}, x_{q(n)})\} p_b(x_{p(n)}, x_{q(n)})$$

$$\leq \alpha_1 \left(p_b(x_{p(n)}, x_{q(n)})\right)^2 + \alpha_2 p_b(x_{p(n)}, x_{q(n)})[2s\, p_b(x_{p(n)}, x_{q(n)}) + p_b(x_{p(n)}, Tx_{p(n)})]$$

$$+\alpha_3 p_b(x_{n(p)}, Tx_{q(n)})[2s\, p_b(x_{p(n)}, x_{q(n)}) + p_b(x_{q(n)}, Tx_{q(n)})] \quad +M\alpha_4 \left(p_b(x_{p(n)}, x_{q(n)})\right)^2$$

$$= (\alpha_1 + 2s\alpha_2 + 2s\, \alpha_3 + M\alpha_4) \left(p_b(x_{p(n)}, x_{q(n)})\right)^2 + \alpha_2\, p_b(x_{p(n)}, x_{q(n)}) p_b(x_{p(n)}, Tx_{p(n)})$$

$$+\alpha_3 p_b(x_{n(p)}, Tx_{q(n)}) p_b(x_{q(n)}, Tx_{q(n)})$$

$$p_b(x_{p(n)}, x_{q(n)}) p_b(Tx_{p(n)}, Tx_{q(n)}) \leq \frac{1}{2s^2} \left(p_b(x_{p(n)}, x_{q(n)})\right)^2$$
$$+ \tfrac{1}{2s^2} p_b(x_{p(n)}, x_{q(n)}) p_b(x_{p(n)}, Tx_{p(n)}) + \tfrac{1}{2s^2} p_b(x_{n(p)}, Tx_{q(n)}) p_b(x_{q(n)}, Tx_{q(n)})$$

$$\Rightarrow \varepsilon^2 \leq \limsup_{n\to\infty} p_b(x_{p(n)}, x_{q(n)}) p_b(Tx_{p(n)}, Tx_{q(n)}) = \limsup_{n\to\infty} p_b(x_{p(n)}, x_{q(n)}) p_b(x_{p(n)+1}, x_{q(n)+1})$$
$$\leq \frac{1}{2s^2} [\left(p_b(x_{p(n)}, x_{q(n)})\right)^2 + p_b(x_{p(n)}, x_{q(n)}) p_b(x_{p(n)}, x_{p(n)+1}) + p_b(x_{n(p)}, Tx_{q(n)}) p_b(x_{q(n)}, x_{q(n)+1})]$$
$$\leq \tfrac{1}{2s^2} (s\varepsilon)^2 = \tfrac{1}{2} (\varepsilon)^2 < \varepsilon^2.$$

This is a contradiction. Hence $p_b(x_n, x_m) = 0$. Since $(X, p_b)$is a complete partial b-metric space, therefore there exists a $x \in X$ such that

$$p_b(x, x) = \lim_{n\to\infty} p_b(x_n, n) = \lim_{n\to\infty} p_b(x_n, x_m) = 0$$

Now we shall prove that $for\ all\ \ n \in N$,

$$\frac{1}{2s} p_b(x_n, Tx_n) < p_b(x_n, x)\ or\ \frac{1}{2s} p_b(Tx_n, T^2x_n) < p_b(Tx_n, x)$$

Again using contradiction, we assume that

$$\frac{1}{2s} p_b(x_n, Tx_n) \geq p_b(x_n, x)\ or\ \frac{1}{2s} p_b(Tx_n, T^2x_n) \geq p_b(Tx_n, x)$$

Therefore

$$\begin{aligned} p_b(x_n, Tx_n) &\leq s[p_b(x_n, x) + p_b(x, Tx_n)] - p_b(x, x) \\ &\leq s[p_b(x_n, x) + p_b(x, Tx_n)] \\ &\leq s\frac{1}{2s}\ p_b(Tx_n, x_n) + s\ \frac{1}{2s} p_b(Tx_n, T^2x_n) \\ &\leq \frac{1}{2}\ p_b(Tx_n, x_n) + \frac{1}{2} p_b(Tx_n, T^2x_n) \\ &< \frac{1}{2}\ p_b(Tx_n, x_n) + \frac{1}{2} p_b(Tx_n, x_n) \\ &<\ p_b(Tx_n, x_n) \end{aligned}$$

This is a contradiction. Hence

$$\frac{1}{2s} p_b(x_n, Tx_n) < p_b(x_n, x)\ or\ \frac{1}{2s} p_b(Tx_n, T^2x_n) < p_b(Tx_n, x)$$

Now we shall prove $\boldsymbol{Tx = x}$. We shall prove it by contradiction**.** Let $\boldsymbol{Tx \neq x}$ **and** $\boldsymbol{P_{n+1}} = p_b(Tx, T^2x_n) > \boldsymbol{P_n} = p_b(x, Tx_n)$. Using the inequality (2.4), we have

$$p_b(Tx_n, x)\ p_b(Tx, T^2x_n) \leq \alpha_1\big(p_b(x, Tx_n)\big)^2 + \alpha_2 p_b(x, Tx_n)[p_b(x, x) + p_b(x, Tx_n)]$$

$$+\alpha_3\ p_b(x, Tx_n)[p_b(Tx_n, Tx_n) + p_b(Tx_n, T^2x_n)]$$

$$+\alpha_4 Max\{p_b(x, Tx_n), p_b(x, x), (Tx_n, Tx_n)\} p_b(x, Tx_n)$$

$$\leq \alpha_1\big(p_b(x, Tx_n)\big)^2 + \alpha_2 p_b(x, Tx_n)[2s\ p_b(x, Tx_n) + p_b(x, Tx_n)]$$

$$+\alpha_3\ p_b(x, Tx_n)[2s\ p_b(x, Tx_n) + p_b(Tx_n, T^2x_n)] + M\alpha_4\{p_b(x, Tx_n)\} p_b(x, Tx_n)$$

$$\leq \alpha_1\big(p_b(x, Tx_n)\big)^2 + (2s+1)\alpha_2\ p_b(x, Tx_n) p_b(x, Tx_n)$$

$$+2s\ \alpha_3\ p_b(x, Tx_n) p_b(x, Tx_n) + \alpha_3\ p_b(x, Tx_n) p_b(Tx_n, T^2x_n) + M\alpha_4 p_b(x, Tx_n)\ p_b(x, Tx_n)$$

$$\leq (\alpha_1 + (2s+1) + 2s\ \alpha_3 + +M\alpha_4)\ p_b(x, Tx_n) p_b(x, Tx_n)\ + \alpha_3\ p_b(x, Tx_n) p_b(Tx_n, T^2x_n)$$

$$(1-\alpha_3) p_b(x, Tx_n)\ p_b(Tx_n, T^2x_n) \leq (\alpha_1 + (2s+1) + 2s\ \alpha_3 + M\alpha_4) p_b(x, Tx_n)\ p_b(x, Tx_n)$$

$$(1-\alpha_3) P_n\ P_{n+1} \leq (\alpha_1 + (2s+1) + 2s\ \alpha_3 + M\alpha_4) {P_n}^2$$

$$P_n\ P_{n+1} \leq \frac{(\alpha_1 + (2s+1) + 2s\ \alpha_3 + M\alpha_4)}{(1-\alpha_3)} {P_n}^2$$

$$P_{n+1} \leq \frac{(\alpha_1 + (2s+1) + 2s\ \alpha_3 + M\alpha_4)}{(1-\alpha_3)} P_n$$

Since $\alpha_1 + (2s+1)\alpha_2 + (2s+1)\alpha_3 + \alpha_4 M < 2s\,[\alpha_1 + 2\alpha_2 + 2\,\alpha_3 + \alpha_4 M] < 2s\frac{1}{2s^2}t < \frac{1}{s}t\;\; for\ all\;\; t > 0$ $\Rightarrow \frac{(\alpha_1+(2s+1)\alpha_2+\,2s\alpha_3+\alpha_4 M)}{(1-\alpha_3)} < 1$. Therefore $p_b(Tx_n, T^2x_n) = P_{n+1} \leq\; P_n = p_b(x, Tx_n)\; for\ all\;\; n \in N$. This is a contradiction. Therefore $P_{n+1} = P_n \Rightarrow p_b(Tx_n, T^2x_n) = p_b(x, Tx_n) \Rightarrow x = Tx_n = Tx$. Hence x is a fixed point.

Now we shall prove that **x is unique.** If possible Let u is another fixed point such that $Tx = x\ and\ Tu = u$.

$$p_b(x,u) = \;p_b(Tx,Tu) \leq \alpha_1 p_b(x,Tu) + \alpha_2[p_b(x,x) + p_b(x,Tx)]$$

$$+\alpha_3\;[p_b(u,u) + p_b(u,Tu) + \alpha_4 Max\{p_b(x,u), p_b(x,x), (u,u)\}]$$

$$\leq \alpha_1 p_b(x,u) + \alpha_2[p_b(x,x) + p_b(x,x)] + \alpha_3[p_b(u,u) + p_b(u,u)] + M\alpha_4 p_b(x,u)$$

$$\leq \alpha_1 p_b(x,u) + 4\;\alpha_2\; p_b(x,u) + 4\;\alpha_3\; p_b(x,u) + M\alpha_4 p_b(x,u)$$

$$p_b(x,u) \leq (\alpha_1 + 4s\;\alpha_2 +\; 4s\;\alpha_3\; + M\alpha_4)p_b(x,u)$$

This is a contradiction, since$\alpha_1 + 4s\;\alpha_2 +\; 4s\;\alpha_3\; + M\alpha_4 < 2s\,[\alpha_1 + 2\alpha_2 + 2\;\alpha_3 + \alpha_4 M] < 2s\frac{1}{2s^2}t < \frac{1}{s}t$ $for\ all\;\; t > 0 \Rightarrow x = u$. Hence x is a unique fixed point.

This completes the proof.

Example: let (X, p_b) be a partial b- metric space, where $X = \mathbb{R}$ and $p_b\colon \mathbb{R} \to \mathbb{R}$ is a self map. Define as

$$p_b(x,y) = |x-y|^2 + 3$$

If $x = 4, y = 7, z = \frac{9}{2}\;\; then\; p_b(4,7) = 12, p_b(4,9/2) = \frac{13}{4}, p_b\left(\frac{9}{2},7\right) = \frac{37}{4}, p_b\left(\frac{9}{2},\frac{9}{2}\right) = 3$

$$p_b\left(4,\frac{9}{2}\right) + p_b\left(\frac{9}{2},7\right) - p_b\left(\frac{9}{2},\frac{9}{2}\right) = \frac{13}{4} + \frac{37}{4} - 3 = \frac{19}{2} \nleq 12 = p_b(4,7)$$

Clearly (X, p_b) is not a partial metric space, but

$$2\{p_b\left(4,\frac{9}{2}\right) + p_b\left(\frac{9}{2},7\right)\} - p_b\left(\frac{9}{2},\frac{9}{2}\right) = 2\{\frac{13}{4} + \frac{37}{4}\} - 3 = 22 \geq 12 = p_b(4,7)$$

$\Rightarrow (X, p_b)$ is a partial b- metric space. Therefore satisfies all the conditions of the Theorem 2.1.

3 Conclusion

In this paper, we gave a newly fixed point theorems for Partial b-metric space. We hope that our study contributes to the development of these results by other researchers.

Acknowledgement

The authors wish to thank the referees for their careful reading of the manuscript and suggestions.

Competing Interests

Authors have declared that no competing interests exist.

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